Submersions, Immersions, and Embeddings

- Because the **pushforward** of a smooth map represents the "best approximation" to a map near a given point, we can learn a great deal about the map itself by studying linear-algebraic properties of its pushforward at each point.
- The most important such property is its **rank** (the dimension of its image).
- The maps for which **pushforwards** give good local models turn out to be the ones whose pushforwards have **constant ranks**.
- Three special categories of such maps will play particularly important roles:
  —submersions(smooth maps whose pushforwards are surjective)
  —immersions(smooth maps whose pushforwards are injective)
  —embeddings(injective immersions that are homepmorphisms onto images.)
- Under appropriate hypotheses on its rank, **a smooth map behaves locally like its pushforward**.

- In linear algebra the **rank** of $m \times n$ matrix A is defined in three equivalent ways:
  (1) the dimension of the subspace of $\mathbb{V}^n$ spanned by the rows,
  (2) the dimension of the subspace of $\mathbb{V}^m$ spanned by the columns,
  (3) the maximum order of any nonvanishing minor of determinant.

Definition. *The* **rank** *of a linear transformation is defined to be the dimension of the image, and this is the rank of any matrix which represents the transformation.*

Definition. *When $F : U \to \mathbb{R}^m$ is a C^1 mapping of an open set $U \subset \mathbb{R}^n$, we refer to the rank of $DF(x)$ as the rank of F at x.*

Example. $F(x^1, x^2) = ((x^1)^2 + (x^2)^2, 2x^1x^2)$ has Jacobian

$$DF(x_1, x_2) = \begin{pmatrix} 2x^1 & 2x^2 \\ 2x^2 & 2x^1 \end{pmatrix}$$

whose rank is 2 on all of $\mathbb{R}^2$ except the lines $x_2 = \pm x^1$. The rank is 1 on these lines except at $(0, 0)$ where it is zero.

Rank Theorem. *Suppose $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets and $F : U \to V$ is a smooth map with constant rank k. For any $p \in U$, there exist smooth coordinate charts (U_0, φ) for (U_0, φ) for $\mathbb{R}^m$ centered at p and (V_0, ψ) for $\mathbb{R}^n$, with $U_0 \subset U$ and $F(U_0) \subset V_0 \subset V$, such that*

$$\psi \circ F \circ \varphi^{-1}(x_1, \cdots, x^k, x^{k+1}, \cdots, x^m) = (x^1, \cdots, x^k, 0, \cdots, 0).$$

Definition 1.15. *A smooth map $g : \mathbb{R}^n \to \mathbb{R}^p$ is a* **submersion** *(resp. an* **immersion**, *resp. a* **local diffeomorphism***) at $x \in \mathbb{R}^n$ if its differential map D_xg at x is surjective (resp. injective, resp. bijective) from $\mathbb{R}^n$ to $\mathbb{R}^p$.*

Proposition 1.17 (canonical form for immersions and submersions). *Let $p \leq n$ be two integers.*

(1) *Let $f : \mathbb{R}^n \to \mathbb{R}^p$ with $f(0) = 0$ be a submersion at 0. Then there exists a local diffeomprphism ϕ around 0 in $\mathbb{R}^n$, with $\phi(0) = 0$ and such that*

$$f \circ \phi(x_1, \cdots, x_n) = (x_1, \cdots, x_p).$$

(2) *Let $f : \mathbb{R}^p \to \mathbb{R}^n$ with $f(0) = 0$ be a immersion at 0. Then there exists a local diffeomprphism ψ around 0 in $\mathbb{R}^n$, with $\psi(0) = 0$ and such that*

$$\psi \circ f(x_1, \cdots, x_p) = (x_1, \cdots, x_p, 0, \cdots, 0).$$

Inverse Function Theorem. *Suppose U and V are open subsets of $\mathbb{R}^n$, and $F : U \to V$ is a smooth map.*
If $DF(p)$ is nonsingular at some point $p \in U$, then there exist connected nbhds $U_0 \subset U$ at p and $V_0 \subset V$ of $F(p)$ such that $F\Big|_{U_0} : U_0 \to V_0$ is a diffeomorphism.

Implicit Function Theorem. *Let $U \subset \mathbb{R}^n \times \mathbb{R}^k$ be an open set, and let $(x, y) = (x^1, \cdots, x^n, y^1, \cdots, y^k)$ denote the standard coordinates on U.*
Suppose $\Phi : U \to \mathbb{R}^k$ is a smooth map, $(a, b) \in U$ and $c = \Phi(a, b)$.
If the $k \times k$ matrix

$$\left(\frac{\partial \Phi^i}{\partial y^j}(a, b)\right)$$

is nonsingular, then there exist nbhds $V_0 \subset \mathbb{R}^n$ of a and $W_0 \subset \mathbb{R}^k$ of b and a smooth map

$$F : V_0 \to W_0$$

such that $\Phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of F; i.e.

$$\Phi(x, y) = c \ \text{ for } (x, y) \in V_0 \times W_0 \Leftrightarrow y = F(x).$$

Rank Theorem. *Suppose $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets and $F : U \to V$ is a smooth map with constant rank k. For any $p \in U$, there exist smooth coordinate charts (U_0, φ) for (U_0, φ) for $\mathbb{R}^m$ centered at p and (V_0, ψ) for $\mathbb{R}^n$, with $U_0 \subset U$ and $F(U_0) \subset V_0 \subset V$, such that*

$$\psi \circ F \circ \varphi^{-1}(x_1, \cdots, x^k, x^{k+1}, \cdots, x^m) = (x^1, \cdots, x^k, 0, \cdots, 0).$$

Proof 1. The fact that $DF(p)$ has rank k implies that its matrix has some $k \times k$ minor with nonzero determinant. By ordering the coordinates, we may assume that it is the upper left minor, $(\partial F^i/\partial x^j)$ for $i, j = 1, \cdots, k$.

• Let us relabel the standard coordinates as

$$(x, y) = (x^1, \cdots, x^k, y^1, \cdots, y^{m-k}) \ \text{ in } \mathbb{R}^m,$$

and

$$(v, w) = (v^1, \cdots, v^k, w^1, \cdots, w^{n-k}) \ \text{ in } \mathbb{R}^n.$$

— By an initial translation of the coordinates, we may assume w.l.o.g. that

$$p = (0, 0) \ \text{ and } \ F(p) = (0, 0).$$

⊙ If we write

$$F(x, y) = (Q(x, y), R(x, y))$$

for some smooth maps $Q : U \to \mathbb{R}^k$ and $R : U \to \mathbb{R}^{n-k}$, then our hypothesis is that $(\partial Q^i/\partial x^j)$ is nonsingular at (0,0).

— Define $\varphi : U \to \mathbb{R}^m$ by

$$\varphi(x, y) = (Q(x, y), y).$$

Its total derivative at $(0,0)$ is

$$D\varphi(0,0) = \begin{pmatrix} \frac{\partial Q^i}{\partial x^j}(0,0) & \frac{\partial Q^i}{\partial y^j}(0,0) \\ 0 & I_{m-k} \end{pmatrix},$$

which is nonsingular because its columns are independent.

— Therefore, by **the inverse function theorem**, there are connected nbhds U_0 of $(0,0)$ and $\widetilde{U}_0$ of $\varphi(0,0)=(0,0)$ such that $\varphi : U_0 \to \widetilde{U}_0$ is a diffeomorphism.

— Writing the inverse map as

$$\varphi^{-1}(x,y) = (A(x,y), B(x,y))$$

for some smooth function $A : \widetilde{U}_0 \to \mathbb{R}^k$ and $B : \widetilde{U}_0 \to \mathbb{R}^{m-k}$, we compute

$$(x,y) = \varphi(A(x,y), B(x,y)) = (Q(A(x,y), B(x,y)), B(x,y)). \tag{7.8}$$

Comparing y components, it follows that $B(x,y) = y$, and therefore φ^{-1} has the form

$$\varphi^{-1}(x,y) = (A(x,y), y).$$

⊙ Observe that $\varphi \circ \varphi^{-1}$ =Id implies $Q(A(x,y),y) = x$, and therefore $F \circ \varphi^{-1}$ has the form

$$F \circ \varphi^{-1}(x,y) = (x, \widetilde{R}(x,y)),$$

where $\widetilde{R} : \widetilde{U}_0 \to \mathbb{R}^k$ is defined by $\widetilde{R}(x,y) = R(A(x,y), y)$.

– The Jacobian matrix of the map at an arbitrary point $(x,y) \in \widetilde{U}_0$ is

$$D(F \circ \varphi^{-1})(x,y) = \begin{pmatrix} I_k & 0 \\ \frac{\partial \widetilde{R}^i}{\partial x^j} & \frac{\partial \widetilde{R}^i}{\partial y^j} \end{pmatrix}.$$

– Since composing with a diffeomorphism does not change the rank of a map, the matrix has rank equal to k everywhere in $\widetilde{U}_0$.

– Since the first k columns are obviously independent, the rank can be k only if the partial derivatives $\frac{\partial \widetilde{R}^i}{\partial y^j}$ vanishes identically on $\widetilde{U}_0$, which implies that $\widetilde{R}$ **is actually independent of** $(y^1, \cdots, y^{m-k})$.

– Thus if we set $S(x) = \widetilde{R}(x,0)$, we have

$$F \circ \varphi^{-1}(x,y) = (x, S(x)). \tag{7.9}$$

⊙ To complete the proof, we need to **define a smooth chart for** $\mathbb{R}^n$ **near** $(0,0)$.

– Let $V_0 \subset V$ be the open set

$$V_0 = \{(v,w) \in V : (v,0) \in \widetilde{U}_0\},$$

which is a nbhd of $(0,0)$ because $(0,0) \in \widetilde{U}_0$, and define

$$\psi : V_0 \to \mathbb{R}^n$$

by

$$\psi(v,w) = (v, w - S(v)).$$

This is a **diffeomorphism** onto its image, because its inverse is given explicitly by

$$\psi^{-1}(s,t) = (s, t + S(s));$$

thus (V_0, ψ) **is a smooth chart**.

– It follows from (7.9) that

$$\psi \circ F \circ \varphi^{-1}(x,y) = \psi(x, S(x)) = (x, S(x) - S(x)) = (x, 0),$$

which was to be proved. □

Proof 2 (k=m). This follows from the implicit function theorem.

— In local coordinates $(z^1, \cdots, z^n)$ on $\mathbb{R}^n$, $(x^1, \cdots, x^m)$ on $\mathbb{R}^m$, let, w.l.o.g. (since $df(x)$ is injective)

$$\left(\frac{\partial z^\alpha(F(x))}{\partial x^i}\right)_{i,\alpha=1,\cdots,m}$$

be nonsingular.

— We consider

$$\widetilde{F}(x,z) = (z^1 - F^1(x), \cdots, z^n - F^n(x)),$$

which has maximal rank in $x^1, \cdots, x^m, z^{m+1}, \cdots, z^n$.

— By the implicit function theorem, there locally exists a map

$$(z^1, \cdots, z^m) \mapsto (\varphi^1(z^1, \cdots, z^m), \cdots, \varphi^n(z^1, \cdots, z^m))$$

with

$$\begin{aligned}\widetilde{F}(x,z) = 0 \Longleftrightarrow & x^1 = \varphi^1(z^1, \cdots, z^m), \cdots, x^m = \varphi^m(z^1, \cdots, z^m), \\ & z^{m+1} = \varphi^{m+1}(z^1, \cdots, z^m), \cdots, z^n = \varphi^n(z^1, \cdots, z^m),\end{aligned}$$

for which $(\frac{\partial \varphi^i}{\partial z^\alpha})_{i,\alpha=1,\cdots,m}$ has maximal rank.

— As new coordinates, we now choose

$$\begin{aligned}(y^1, \cdots, y^n) =& (\varphi^1(z^1, \cdots, z^m), \cdots, \varphi^m(z^1, \cdots, z^m), \\ & z^{m+1} - \varphi^{m+1}(z^1, \cdots, z^m), \cdots, z^n - \varphi^n(z^1, \cdots, z^m))\end{aligned}$$

Then

$$\begin{aligned}z = f(x) \Leftrightarrow & \widetilde{F}(x,z) = 0 \\ \Leftrightarrow & (y^1, \cdots, y^n) = (x^1, \cdots, x^m, 0. \cdots, 0),\end{aligned}$$

and the claim follows. □

Constant-Rank Maps Between Manifolds

- Let M and N be a C^p manifolds. Let $F : M \to N$ be C^k for $k \leq p$. If (U, φ) and (V, ψ) are charts for M and N around p and $F(p)$ respectively, with $F(U) \subset V$, then let
$$\hat{F} = \psi \circ F \circ \phi^{-1} : \varphi(U) \to \psi(V)$$

Definition. *The* **rank** *of F at $p \in M$ is defined to be the rank of $\hat{F} = \psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$, which is the rank at $\varphi(p)$ of the Jacobian matrix of the mapping $\hat{F}(x^1, \cdots, x^n) = (f^1(x^1, \cdots, x^n), \cdots, f^m(x^1, \cdots, x^n))$*
- *If F has the same rank k at every point, we say that it has* **constant rank**, *and write $\operatorname{rank} F = k$.*

- The rank of $F : M \to N$ at $p \in M$ is the rank of the linear map $F_* : T_m M \to T_{F(p)} N$, which is the dimension of $\mathrm{Im} F_* \subset T_{F(p)} N$.

Definition 1.16. *Let M and N be smooth manifolds. A smooth map $f : M \to N$ is an* **immersion** *at $m \in M$ if for a chart (U, ϕ) for M around m and a chart (V, ψ) around $f(m)$, the map $\psi \circ f \circ \phi^{-1}$ is itself an immersion.*
- *This definition does not depend on the chart.*

Definition 1.18. *A map $f : M \to N$ is a* **submersion** *(resp. an* **immersion**, *resp. a* **local diffeomorphism***) if it has this property at any point of M.*
— *The map f is a* **diffeomorphism** *if it is bijective, and if f and f^{-1} are smooth.*
— *The map f is an* **embedding** *if it is an immersion and if it is a homeomorphism on its image.*

Definition. *If $F : M \to N$ is a smooth map, we define the* **rank** *of F at $p \in M$ to be the rank of the linear map*
$$F_* : T_p M \to T_{F(p)} N;$$
it is the rank of the matrix of partial derivatives of F in any smooth chart, or the dimension of $\mathrm{Im} F_* \subset T_{F(p)} N$.
- *If F has the same rank k at every point, we say that it has* **constant rank**, *and write $\operatorname{rank} F = k$.*

Definition. *A smooth map $F : M \to N$ of constant rank is called a* **submersion** *if F_* is surjective at each point (or equivalently, if rank F=dim N).*
- *It is called an* **immersion** *if F_* is injective at each point (or equivalently, if rank F=dim M).*
- *A* **smooth embedding** *is an immersion $F : M \to N$ that is also a topological embedding, i.e. a homeomorphism onto its image $F(M) \subset N$ in the subspace topology.*

- Notice that although submersions and immersions are smooth maps by definition, there are two types of embeddings, topological and smooth.

— A smooth embedding is a map that is both a topological embedding and an immersion, not just a topological embedding that happens to be smooth.

Rank Theorem for Manifolds. *Suppose M and N are smooth manifolds of dimension m and n, respectively, and $F : M \to N$ is a smooth map with constant rank k. For each $p \in M$, there exist smooth coordinates $(x^1, \cdots, x^k, x^{k+1}, \cdots, x^m)$ centered at p and $(v^1, \cdots v^n)$ centered at $F(p)$ in which F has the coordinate representations*

$$F(x^1, \cdots, x^k, x^{k+1}, \cdots, x^m) = (x^1, \cdots, x^k, 0, \cdots, 0). \tag{1}$$

Proof. Replacing M and N by smooth coordinate domains $U \subset M$ near p and $V \subset N$ near $F(p)$ and replacing F by its coordinate representation, the theorem is reduced to the rank theorem in Euclidean space. □

Inverse Function Theorem for Manifolds. *Suppose M and N are smooth manifolds, $p \in M$, and $F : M \to N$ is a smooth map such that $F_* : T_pM \to T_{F(p)}N$ is bijective. Then there exist connected nbhds U_0 of p and V_0 of $F(p)$ such that $F\big|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Proof. The fact that F_* is bijective implies that M and N have the same dimension, and then the result follows from the Euclidean inverse function theorem applied to the coordinate representation of F. □

Corollary 11. *Suppose M and N are smooth manifold of the same dimension, and $F : M \to N$ is an immersion or submersion. Then F is a local diffeomorphism. If F is bijective, it is a diffeomorphism.*

Emebdded Submanifolds

- Smooth submanifolds are modeled locally on the standard embeddings of $\mathbb{R}^k$ into $\mathbb{R}^n$, identifying $\mathbb{R}^k$ with the subspace

$$\{(x^1, \cdots, x^k, x^{k+1}, \cdots, x^n) : x^{k+1} = \cdots = x^n = 0\}$$

of $\mathbb{R}^n$. Somewhat more generally, if U is an open subset of $\mathbb{R}^n$, a k-**slice** of U is any subset of the form

$$S = \{(x^1, \cdots, x^k, x^{k+1}, \cdots, x^n) \in U : x^{k+1} = c^{k+1}, \cdots, x^n = c^n\}$$

for some constants $c^{k+1}, \cdots, c^n$. Clearly, any k-slice is homeomorphic to an open subset of $\mathbb{R}^k$.

Definition. *Let M be a smooth n-manifold, and (U, φ) be a smooth chart on M.*

- *If S is a subset of U such that $\varphi(S)$ is a k-slice of $\varphi(U)$, then we say simply that S is a k-**slice** of U.*
- *A subset $S \subset M$ is called an **embedded submanifold of dimension** k (or **embedded k-submanifold** for short) if $\forall p \in S$, $\exists$a smooth chart (U, φ) for M such that $p \in U$ and $U \cap S$ is a k-slice of U.*

⊙ *In this situation we call the chart (U, φ) a **slice chart** for S in M, and the corresponding coordinates $(x_1, \cdots, x_n)$ are called **slice coordinates**.*

Definition. *If S is an embedded submanifold of M, the difference $\dim M - \dim S$ is called the **codimension of** S **in** M.*

- *An **embedded hypersurface** is an embedded submanifold of codimension* 1.

Definition 1.1. *A subset $M \subset \mathbb{R}^{n+k}$ is an n-**dimensional submanifold of class** C^p of $\mathbb{R}^{n+k}$ if, $\forall x \in M$, $\exists$a nbhd U of x in $\mathbb{R}^{n+k}$ and a C^p submersion $f : U \to \mathbb{R}^k$ such that $U \cap M = f^{-1}(0)$.*

Definition 1.9. *Let M be a smooth manifold of dimension d. A subset $N \subset M$ is a **submanifold** of M if for any $p \in N$, there exists a chart (U, ϕ) of M around p such that $\phi(U \cap N)$ is a submanifold of the open set $\phi(U) \cap \mathbb{R}^d$.*

Examples of Embedded Submanifolds

Lemma 6 (Graphs as Submanifolds). *If $U \subset \mathbb{R}^n$ is open and $F : U \to \mathbb{R}^k$ is smooth, then the graph of F is an embedded n-dimensional submfd of $\mathbb{R}^{n+k}$.*

Proof. Define a map $\varphi : U \times \mathbb{R}^k \to U \times \mathbb{R}^k$ by

$$\varphi(x, y) = (x - F(x)).$$

It is clearly smooth, and in fact it is a diffeomorphism because its inverse can be written explicitly:

$$\varphi^{-1}(u, v) = (u, v + F(u)).$$

Because $\varphi(\Gamma(F))$ is the slice $\{(u, v) : v = 0\}$ of $U \times \mathbb{R}^k$, this shows that $\Gamma(F)$ is an embedded submanifold. □

Example 7 (Spheres). $\forall n \geq 0$, $\mathbb{S}^n$ is an embedded submfd of $\mathbb{R}^{n+1}$, because it is locally the graph of a smooth function.

Proposition 1.3. *The following are equivalent:*

(1) *M is a C^p submanifold of dimension n of $\mathbb{R}^{n+k}$.*
(2) *$\forall x \in M$, $\exists$a nbhd U of x in $\mathbb{R}^{n+k}$ and a C^p submersion $f : U \to \mathbb{R}^k$ such that $U \cap M = f^{-1}(0)$.*
(3) *$\forall x \in M$, $\exists$open nbhds U and V of x and 0 in $\mathbb{R}^{n+k}$ respectively, and a C^p diffeomorphism $f : U \to V$ such that*

$$f(U \cap M) = V \cap (\mathbb{R}^n \times \{0\}).$$

(4) *$\forall x \in M$, $\exists$nbhd U of x in $\mathbb{R}^{n+k}$, a nbhd Ω of O in $\mathbb{R}^n$, and a C^p map $g : \Omega \to \mathbb{R}^{n+k}$ such that (Ω, g) is a local parametrization of $M \cap U$ around x (that is g is an homeomorphism from Ω onto $M \cap U$ and $g'(O)$ is injective.)*

- The implicit function theorem implies the following.

Lemma 1.3.2. *Let $f : M \to N$ be a differentiable map, $\dim M = m$, $\dim N = n$, $m \geq n$, $p \in N$. Let $df(x)$ have rank n for all $x \in M$ with $f(x) = p$. Then $f^{-1}(p)$ is a smooth submanifold of M of dimension $m - n$.*

Proof. Again representation the situation in local coordinates around $x \in M$ and $p = f(x) \in N$. In these coordinates $df(x)$ still has rank n.

— By the implicit function theorem, there exist an open nbhd U of x and a differentiable map

$$g(x^{n+1}, \cdots, x^m) : U_2 \subset \mathbb{R}^{m-n} \to U_1 \subset \mathbb{R}^n$$

with

$$U = U_1 \times U_2$$

and

$$f(x) = p \Longleftrightarrow (x^1, \cdots, x^n) = g(x^{n+1}, \cdots, x^m).$$

— With

$$\begin{aligned}(y^1, \cdots, y^n) =& (x^1, \cdots, x^n) - g(x^{n+1}, \cdots, x^n) \\ (y^{n+1}, \cdots, y^m) =& (x^{n+1}, \cdots, x^m).\end{aligned}$$

We then obtain coodinates for which

$$f(x) = p \Longleftrightarrow y^\alpha = 0, \quad \text{for } \alpha = 1, \cdots, n.$$

$(y^{n+1}, \cdots, y^m)$ thus yield local coordinates for $\{f(x) = p\}$
and this implies that $\{f(x) = p\}$ is a submanifold of dimension $m - n$. □

Level Sets

Definition. *If* $\Phi : M \to N$ *is any map and* c *is any point of* N*, the set* $\Phi^{-1}(c)$ *is called a* **level set** *of* Φ*.*

Example. The n-sphere $S^n \subset \mathbb{R}^{n+1}$ is the level set $\Phi^{-1}(1)$, where $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}$ is the function $\Phi(x) = |x|^2$.

Example. Consider the two maps Φ, $\Psi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\Phi(x, y) = x^3 - y^2, \quad \text{and} \quad \Psi(x, y) = x^2 - y^2.$$

— The zero set of Φ is a curve that has a "cusp", or "kink" at the origin, while the zero set of Ψ is the union of the lines $x = y$ and $x = -y$.
— Neither of these sets is an embedded submanifolds of $\mathbb{R}^2$.

- To give some general criteria for level sets to be submanifolds, consider first a linear version of the problem.

⊙ Any k-dimensional linear subspace $S \subset \mathbb{R}^n$ is the kernel of some linear map. (Such a linear map is easily constructed by choosing a basis for S and extending it to a basis for $\mathbb{R}^n$.)
⊙ By the rank-nullity law, if $S = \ker L$, then $\mathrm{Im} L$ must have dimension $n - k$.
⊙ Therefore, a natural way to specify a k-dimensional subspace $S \subset \mathbb{R}^n$ is to give a surjective linear map $L : \mathbb{R}^n \to \mathbb{R}^{n-k}$ whose kernel is S.
— The vector equation $Lx = 0$ is equivalent to $n - k$ independent scalar equation, each of which can be thought of as cutting down one of the degrees of freedom in $\mathbb{R}^n$, leaving a subspace S of dimension k.

- In the context of smooth manifolds, the analogue of a surjrctive linear map is a submersion.

— Thus we might expect that **a level set of a submersion from an n-manifold to an $(n - k)$-manifold** should be an embedded k-dimensional submanifold.

Constant-Rank Level Set Theorem. *Let* M *and* N *be smooth manifolds, and let* $\Phi : M \to N$ *be a smooth map with constant rank to* k*. Each level set of* Φ *is a closed embedded submanifold of codimension* k *in* M*.*

Proof. Let $c \in N$ be arbitrary, and let S denote the level set $\Phi^{-1}(c) \subset M$.
Clearly, S is closed in M by continuity.
— To show that is an embedded submanifold, we need to show that $\forall p \in S$, there is a slice chart for S in M near p.
— From the rank theorem, there are smooth charts (U, φ) centered at p and (V, ψ) centered at $c = \Phi(p)$ in which Φ has a coordinate representation of the form

$$\Phi(x^1, \cdots, x^k, x^{k+1}, \cdots, x^m) = (x^1, \cdots, x^k, 0, \cdots, 0),$$

and therefore $S \cap U$ is the slice $\{(x^1, \cdots, x^m) \in U : x^1 = \cdots x^k = 0\}$. □

Submersion Level Set Theorem. *If* $\Phi : M \to N$ *is a submersion, then each level set of* Φ *is a closed embedded submanifold whose codimension is equal to the dimension of* N*.*

Proof. A submersion has constant rank equal to the dimension of N. □

Definition. *If $\Phi : M \to N$ is a smooth map, a point $p \in M$ is said to be a* **regular point** *of Φ if $\Phi_* : T_pM \to T_{\Phi(p)}N$ is surjective; it is a* **critical point** *otherwise.*

- *A point $c \in N$ is said to be a* **regular value** *of Φ if every point of the level set $\Phi^{-1}(c)$ is a regular point, and a* **critical value** *otherwise.*
- *A level set $\Phi^{-1}(c)$ is called a* **regular level set** *if c is a regular value; in other words, a regular level set is a level set consisting entirely of regular points.*

Regular Level Set Theorem. *Every regular level set of a smooth map is a closed embedded submanifold whose codimension is equal to the dimension of the range.*

Proof. Let $\Phi : M \to N$ be a smooth map and let $c \in N$ be a regular value such that $\Phi^{-1}(c) \neq \phi$.

— The fact that c is a regular value means that ϕ_* has rank equal to the dimension of N at every point of $\Phi^{-1}(c)$.

— It suffices to show that the set U of points where rankΦ_* =dim$N$ is **open** in M, for then $\phi\Big|_U : U \to N$ is a submersion, and we can apply the preceeding orollary with M replaced by U, noting that an embedded submfd of U is also an embedded submfd of M.

- **To see that U is open**, let m =dim$M$, n=dim$N$, and suppose $p \in U$.

— In terms of smooth coordinates near p and $\Phi(p)$, the assumption that rank$\Phi_* = n$ at p means that the $n \times m$ matrix representing Φ_* in coordinates has an $n \times n$ minor whose determinant is nonzero.

— By continuity, the determinant will be nonzero in some nbhd of p, which means that Φ has rank n in this whole nbhd. □

Example (Spheres). Now we can give a much easier proof that $\mathbb{S}^n$ is an embedded n-dimensional submfd of $\mathbb{R}^{n+1}$.

— The sphere is easily seen to be a regular level set of the function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $f(x) = |x|^2$, since $df = 2\sum_i x^i dx^i$ vanishes only at the origin.

Proposition 12. *Let S be a subset of a smooth n-manifold M. Then S is an embedded k-submfd of M iff evey point $p \in S$ has a nbhd U in M such thst $U \cap S$ is a level set of a submersion $\Phi : U \to \mathbb{R}^{n-k}$.*

Proof. (⇒)Suppose S is an embedded k-submfd.

— If $(x^1, \cdots, x^n)$ are slice coordinates for S on an open set $U \subset M$, the map $\Phi : U \to \mathbb{R}^{n-k}$ given the coordinates by $\Phi(x) = (x^{k+1}, \cdots, x^n)$ is easily seen to be a submersion one of whose level sets is $S \cap U$.

(⇐) Suppose that around every point $p \in S$ there is a nbhd U and a submersion $\Phi : U \to \mathbb{R}^{n-k}$ such that $S \cap U = \Phi^{-1}(c)$ for some $c \in \mathbb{R}^{n-k}$.

— By the submersion level set theorem, $S \cap U$ is an embedded submfd of U.

— Hence, S is itself an embedded submfd. □

The Tangent Space to an Embedded Submanifold

- If S is an embedded submanifold of $\mathbb{R}^n$, we intuitively think of the tangent space T_pS at a point of S as a subspace of the tangent space $T_p\mathbb{R}^n$.
- Similarly, the tangent space to a submanifold of an abstract manifold can be viewed as a subspace of the tangent space to the ambient manifold, once we make appropriate identifications.

- Let M be a smooth manifold and $S \subset M$ be an embedded submanifold.

— Since the inclusion map $\imath : S \hookrightarrow M$ is a smooth embedding, at each point $p \in S$ we have an injective linear map

$$\imath_* : T_pS \to T_pM.$$

In terms of derivations, this injection works in the following way:
For any vector $X \in T_pS$, the image vector $\widetilde{X} = \imath_* X \in T_pM$ acts on smooth functions on M by

$$\widetilde{X}f = (\imath_* X)f = X(f \circ \imath) = X(f\Big|_S).$$

We will adopt the convention of **identifying** T_pS with its image under this map, thereby thinking of T_pS as a certain linear subspace of T_pM.

Lemma 1.3.3. *In the situation of Lemma 1.3.2, we have for the submanifold $S = f^{-1}(p)$ and $q \in X$*

$$T_qS = \ker df(q) \subset T_qM.$$

Proof. Let $v \in T_\varphi X$, (U, φ) a chart on S with $q \in U$. Let γ be any smooth curve in $\varphi(U)$ with

$$\gamma(0) = \varphi(q), \quad \gamma'(0) = \frac{d}{dt}\gamma(t)\Big|_{t=0} = \varphi_* v,$$

for example, $\gamma(t) = \varphi(q) + t\varphi_* v$.

— Then $c = \varphi^{-1}(\gamma)$ is a curve in S with $c'(0) = v$.

— Because $S = f^{-1}(p)$, $f \circ c(t) = p, \quad \forall t$, hence

$$df(q) \circ c'(0) = 0,$$

and therefore $v = c'(0) \in \ker df(q)$.

— Since also $\dim T_qS = m - n =$dim $\ker df(q)$, the claim follows. □

Example. For our example $\mathbb{S}^n$, we may choose

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, \quad f(x) = |x|^2.$$

Then

$$T_x\mathbb{S}^n = \ker df(x) = \{v \in \mathbb{R}^{n+1} : x \cdot v = 0\}$$

Proposition 5. *Suppose $S \subset M$ is an embedded submanifold and $p \in S$. As a subspace of T_pM, the tangent space T_pS is given by*

$$T_pS = \{X \in T_pM : Xf = 0 \ \ \textit{whenever } f \in C^\infty(M) \textit{ and } f\Big|_S \equiv 0\}.$$

Proof. **(I)** Fist suppose $X \in T_pS \subset T_pM$. This means, more precisely, that

$$X = \imath_* Y$$

for some $Y \in T_pS$. If f is any smooth real-valued function on M that vanishes on S, then $f \circ \imath \equiv 0$, so

$$Xf = (\imath_* Y)f = Y(f \circ \imath) \equiv 0.$$

(II) Conversely, if $X \in T_pM$ satisfies $Xf = 0$ whenever f vanishes on S, we need to show that there is a vector $Y \in T_pS$ such that $X = \imath_* Y$.

(i) Let (x^i) be slice coordinates for S in some nbhd U of p, so that $U \cap S$ is the subset of U where $x^{k+1} = \cdots = x^n = 0$, and $(x^1, \cdots, x^k)$ are coordinates for $U \cap S$.

– Because the inclusion map $\imath : U \cap S \hookrightarrow M$ has the coordinate representation

$$\imath(x^1, \cdots, x^k) = (x^1, \cdots, x^k, 0, \cdots, 0)$$

in these coordinates, it follows that T_pS **(that is, $\imath_* T_pS$) is exactly the subspace of T_pM spanned by $\frac{\partial}{\partial x^1}\big|_p, \cdots, \frac{\partial}{\partial x^k}\big|_p$.**

— Writing the coordinate representation of X as

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}\Big|_p,$$

we see that $X \in T_pS$ iff $X^i = 0$ for $i > k$.

— Let φ be smooth bump function supported in U that is equal to 1 in a nbhd of p. Choose an index $j > k$, and consider the function $f(x) = \varphi(x)x^j$, extended to be zero on $M \setminus U$. Then f vanishes identically on S, so

$$0 = Xf = \sum_{i=1}^n X^i \frac{\partial(\varphi(x)x^j)}{\partial x^i}(p) = X^j.$$

Thus $X \in T_pS$, as desired. □