Lie group

Definition. *A collection of elements G together with a binary operation, called $\cdot$, is a group if it satisfies the axiom:*

(1) **Associativity**: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, *if* x, y *and* Z *are in* G.
(2) **Right identity**: G *contains an element* e *such that,* $\forall x \in G$, $x \cdot e = x$.
(3) **Right inverse**: $\forall x \in G$, $\exists$ *an element called* x^{-1}, *also in* G, *for which* $x \cdot x^{-1} = e$.

Definition. *A group is abelian (commutative) of in addtion*

$$x \cdot y = y \cdot x, \quad \forall x, y \in G.$$

Definition. *A* **topological group** *is a topological space with a group structure such that the multiplication and inversion maps are contiunuous.*

Definition. *A* **Lie group** *is a smooth manifold G that is also a group in the algebraic sense, with the property that the multiplication map $m : G \times G \to G$ and inversion map $i : G \to G$, given by*

$$m(g, h) = gh, \quad i(g) = g^{-1},$$

are both smooth.

- A Lie group is, in particular, a topological group.
- It is traditional to denote the identity element of an arbitrary Lie group by the symbol e.

Example 2.7 (Lie Groups).

(a) The **general linear group** $\mathrm{GL}(n, \mathbb{R})$ is the set of invertible $n \times n$ matrices with real entries.
— It is a group under matrix multiplication, and it is an open submanifold of the vector space $M(n, \mathbb{R})$.
— Multiplication is smooth because the matrix entries of a product matrix AB are polynomials in the entries of A and B.
— Inversion is smooth because Cramer's rule expresses the entries of A^{-1} as rational functions of the entries of A.

(b) The **complex general linear group** $\mathrm{GL}(n, \mathbb{C})$ is the set of invertible $n \times n$ matrices with complex entries.
— It is a group under matrix multiplication, and it is an open submanifold of the vector space $M(n, \mathbb{C})$.
— Multiplication and incersion are smooth functions of the real and imaginary parta of the entries of A and B.

(c) If V is any real or complex vector space, we let $\mathrm{GL}(V)$ denote the set of invertible linear transformations from V to itself.
It is a group under composition.

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— If V is finite-dimensional, any basis for V determines an isomorphism of GV(V) with GL($n, \mathbb{R}$) or GL($n, \mathbb{C}$), with n =dim$V$, so GL(V) is a group.
— The trsnsition map between any two such isomorphisms is given by a map of the form $A \mapsto BAB^{-1}$ (where B is the transition matrix between the two bases), which is smooth. Thus the smooth manifold structure on GL(V) is independent of the choice of basis.

(d) The real number field $\mathbb{R}$ and Euclidean space $\mathbb{R}^n$ are Lie groups under addition because the coordinates of $x - y$ are smooth (linear) functions of (x, y).

(e) The set $\mathbb{R}^*$ of nonzero real numbers is a 1-dimensional Lie group under multiplication. (In fact, it is exactly GL($1, \mathbb{R}$) if we identify a 1×1 matrix with the corresponding real number).
— The subset $\mathbb{R}^+$ of positive real numbers is an open subgroup, and is thus itself a 1-dimensional Lie group.

(f) The set $\mathbb{C}^*$ of nonzero complex numbers is a 2-dimensonal Lie group under complex multiplication, which can be identified with GL($1, \mathbb{C}$).

(g) On a subset U of he circle $\mathbb{S}^1 \subset \mathbb{C}^*$, an angle function $\theta : U \to \mathbb{R}$ is a continuous function such that $e^{i\theta(p)} = p$, $\forall p \in U$. For any such angle function θ, $\mathbb{S}^1$ is a smooth manifold with smooth coordinate charts (U, θ).
— $\mathbb{S}^1 \subset \mathbb{C}^*$ is a group under complex multiplication.
— With appropriate angle functions as local coordinates on open subsets of $\mathbb{S}^1$, multiplication and inversion have the smooth coordinate expressions

$$(\theta_1, \theta_2) \mapsto \theta_1 + \theta_2 \quad \text{and} \quad \theta \mapsto -\theta,$$

and therefore $\mathbb{S}^1$ is a Lie group, called the **circle group**.

(h) If $G_1, \cdots, G_k$ are Lie groups, their **direct product** is the product manifold $G_1 \times \cdots \times G_k$ with the group structure given by componentwise multiplication:

$$(g_1, \cdots, g_k)(g'_1, \cdots, g'_k) = (g_1 g'_1, \cdots, g_k g'_k).$$

It is a Lie group, as you can easily check.

(i) The n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is an n-dimensional abelian Lie group.

(j) Any finite or countably infinite group with the diecrete topology is a zero-dimensional Lie group.
— We will call any such group a **discrete group**.

Definition. *If G and H are Lie groups, a* **Lie group homomorphism** *from G to H is smooth map $F : G \to H$ that is also a group homomorphiem.*

- *It is called a* **Lie group isomorphism** *if it is also a diffeomorphism, (which implies that it has an inverse that is also a Lie group homomorphism). In this case we say that G and H are* **isomorphic Lie groups**.

Example 2.8 (Lie Group Homomorphiems).

(a) The inclusion map $\mathbb{S}^1 \hookrightarrow \mathbb{C}^*$ is a Lie group homomorphism.

(b) The map $\exp : \mathbb{R} \to \mathbb{R}^*$ given by $\exp(t) = e^t$ is smooth, and is a Lie group homomorphism because $e^{(s+t)} = e^s e^t$.
(Note that $\mathbb{R}$ is considered as a Lie group under addition, while $\mathbb{R}^*$ is a Lie group under multiplication.)
— The image of exp is the open subgroup $\mathbb{R}^+$ consisting of positive real numbers, and $\exp : \mathbb{R} \to \mathbb{R}^+$ is a Lie group isomorphism with inverse $\log : \mathbb{R}^+ \to \mathbb{R}$.

(c) Similarly, $\exp : \mathbb{C} \to \mathbb{C}^*$ given by $\exp(z) = e^z$ is a Lie group homomorphism.
— It is surjective but not injective, because its kernel consists of the complex numbers of the form $2\pi ik$, where k is an integer.

(d) The map $\varepsilon : \mathbb{R} \to \mathbb{S}^1$ defined by $\varepsilon(t) = e^{2\pi it}$ is a Lie group homomorphism whose kernel is the set $\mathbb{Z}$ of integers.
— Similarly, the map $\varepsilon^n : \mathbb{R}^n \to \mathbb{T}^n$ defined by $\varepsilon^n(t_1, \cdots, t_n) = (e^{2\pi it_1}, \cdots, e^{2\pi it_n})$ is a Lie group homomorphism whose kernel is $\mathbb{Z}^n$.

(e) The determinant function $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$ is smooth because $\det A$ is a polynomial in the matrix entries of A.
— It is a Lie group homomorphism because $\det(AB) = (\det A)(\det B)$.
— Similarly, $\det : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^*$ is a Lie group homomorphism.

(f) If G is any Lie group and $g \in G$, define $C_g : G \to G$ to be conjugation by g:

$$C_g(h) = ghg^{-1}.$$

Then C_g is smooth because group multiplication is smooth, and a simple computation shows that it is a group homomorphism.

Lie Group Actions

- The most important application of Lie groups involve actions by Lie groups on other manifolds.

— This typically arise in situations involving some kind of symmetry.
— For example, if M is a vector space or smooth manifold endowed with a certain geometric structure (such as an inner product, a norm, a metric, or a distinguished vector or covector field), the set of diffeomorphisms of M that preserves this structure (called the **symmetry group** of the structure) frequently turn out to be a Lie group acting smoothly on M.
— The properties of the group action can shed considerable light on the properties of the structure.

Group Actions

Definition. *If G is group and M is a set, a* **left action** *of G on M is a map $G \times M \to M$, often written as $(g, p) \mapsto g \cdot p$ that satisfies*

$$\begin{aligned} g_1 \cdot (g_2 \cdot p) =& (g_1 g_2)p, \quad \forall g_1, g_2 \in G \text{ and } p \in M; \\ e \cdot p =& p, \quad \forall p \in M. \end{aligned} \tag{9.1}$$

- *A* **right action** *is defined analogously as a map $M \times G \to M$ with the appropriate composition law:*

$$\begin{aligned} (p \cdot g_1) \cdot g_2 =& p \cdot (g_1 g_2), \quad \forall g_1, g_2 \in G \text{ and } p \in M; \\ p \cdot e =& p, \quad \forall p \in M. \end{aligned}$$

Definition. *Suppose G is a Lie group and M is a manifold. An action of G on M is said to be* **continuous** *if the map $G \times M \to M$ or $M \times G \to M$ defining the action is continuous.*

- It is useful to give a name to an action, such as $\theta : G \times M \to M$, with the action of a group element g on a point p usually written $\theta_g(p)$.
  In terms of this notation, this condition (9.1) for a left action read

$$\begin{aligned} \theta_{g_1} \circ \theta_{g_2} =& \theta_{g_1 g_2}, \\ \theta_e =& \mathrm{Id}_M, \end{aligned} \tag{9.2}$$

  while for a right action the first equation is replaced by

$$\theta_{g_2} \circ \theta_{g_1} = \theta_{g_1 g_2}.$$

- For a continuous action, each map $\theta_g : M \to M$ is a homeomorphism, because $\theta_{g^{-1}}$ is a continuous inverse for it.
  If the action is smooth, then each θ_g is a diffeomorphism.

Definition. *Define a* **global flow** *on M (sometimes also called a* **one-parameter group action***) to be a left action of $\mathbb{R}$ on M; that is, a continuous map $\theta : \mathbb{R} \times M \to M$ satisfying the following properties $\forall s, t \in \mathbb{R}$ and $\forall p \in M$:*

$$\theta(t, \theta(s, p)) = \theta(t + s, p), \quad \theta(0, p) = p.$$

Example. Let V be a smooth vector field on a smooth manifold M with the property that $\forall p \in M$ there is a unique integral curve $\theta^{(p)} : \mathbb{R} \to M$ starting at p.
- $\forall t \in \mathbb{R}$, we can define a map θ_t from M to itself by sending each point $p \in M$ to the point obtained by following the integral curve starting at p for time t:

$$\theta_t(p) = \theta^{(p)}(t).$$

  This defines a family of maps $\theta_t : M \to M$ for $t \in \mathbb{R}$.
- — Each such map **"slides" the entire manifold along the integral curves** for time t, and it satisfies

$$\theta_t \circ \theta_s(p) = \theta_{t+s}(p).$$

- — Together with the equation $\theta_0(p) = \theta^{(p)}(0) = p$, which holds by definition, this implies that the map $\theta : \mathbb{R} \times M \to M$ is an action of the additive group $\mathbb{R}$ on M.

Definition. *Suppose $\theta : G \times M \to M$ is a left action of a group G on a set M. (The definitions for right actions are analogous.)*

- *For any $p \in M$, the* **orbit** *of p under the action is the set*

$$G \cdot p = \{g \cdot p : g \in G\},$$

  the set of all images of p under the action of elements of G.
- *The action is* **transitive** *if for any two points $p, q \in M$, there is a group element g such that $g \cdot p = q$, or equivalently if the orbit of any point is all of M.*
- *Given $p \in M$, the* **isotropy group** *of p, denoted by G_p, is the set of elements $g \in G$ that fixes p:*

$$G_p = \{g \in G : g \cdot p = p\}$$

- *The action is said to be* **free** *if the only element of G that fixes any point of M is the identity:*

$$g \cdot p = p \quad \text{for some } p \in M. \Rightarrow g = e.$$

  This is equivalent to the requirement that $G_p = \{e\}$ for every $p \in M$.

Examples (Lie Group Actions).

(a) If G is any Lie group and M is any smooth manifold, the **trivial action** of G on M is defined by

$$g \cdot p = p, \quad \forall g \in G.$$

It is easy to see that it is a smooth action, and the isotropy group of each point is all of G.

(b) The naturality action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ is left action given by matrix multiplication:

$$(A, x) \mapsto Ax,$$

considering $x \in \mathbb{R}^n$ as a column matrix.

— This is an action because $I_n x = x$ and matrix multiplication is associative: $(AB)x = A(Bx)$.

— It is smooth because the components of Ax depend polynomially on the matrix entries of A and the components of x.

— Because any nonzero vector can be taken to be any other by some linear transformation, there are **exactly two orbits:** $\{0\}$ **and** $\mathbb{R}^n \setminus \{0\}$.

(c) The restriction of the natural action to $O(n) \times \mathbb{R}^n \to \mathbb{R}^n$ definies a smooth left action of $O(n)$ on $\mathbb{R}^n$.

— In this case, the orbits are the origin and **the spheres centered at the origin**.

(d) Further restricting the natural action $O(n) \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, we obtain a transitive action of $O(n)$ on $\mathbb{S}^{n-1}$.

(e) The natural action of $O(n)$ restricts to a smooth action of $SO(n)$ on $\mathbb{S}^{n-1}$.

— When $n = 1$, the action is trivial because $SO(1)$ is the trivial group consisting of the matrix (1).

— When $n > 1$, $SO(n)$ acts transitively on $\mathbb{S}^{n-1}$.

(f) Any Lie group G acts smoothly, freely, and transitively on itself by left or right translation.
— More generally, if H is Lie subgroup of G, then the restrction of the multiplication map to $H \times G \to G$ defines a smooth, free (but generally, not transitive) left action of H on G.
— Similarly, restriction to $G \times H \to G$ defines a free right action of H on G.

(g) Similarly, any Lie group acts smoothly on itself by **conjugation**:

$$g \cdot h = ghg^{-1}.$$

For any $h \in G$, the isotropy group G_h is the set of all elements of G that commutes with h.

(h) An action of a discrete group Γ on a manifold M is smooth iff for each $g \in \Gamma$, the map $g \mapsto g \cdot p$ is a smooth map from M to itself.
— Thus, for example, $\mathbb{Z}^n$ acts smoothly on the left on $\mathbb{R}^n$ by translation:

$$(m^1, \cdots, m^n) \cdot (x^1, \cdots, x^n) = (m^1 + x^1, \cdots, m^n + x^n).$$