Representations

Definition. *A* **realization** *of a group G is a map between any element g of G and a transformation $T(g)$ of some space M in such a way that the group properties are preserved:*

1. $\rho(e) = Id$*, the identity transformation (no change of* M*);*
2. $\rho(g^{-1}) = [\rho(g)]^{-1}$*;*
3. $\rho(g) \circ \rho(h) = \rho(gh)$*.*

- *The realization is* **faithful** *if the map is injective:*

$$\rho(g) \neq \rho(h) \quad \text{if } g \neq h.$$

- *If M is a vector space and every $\rho(g)$ is a linear transfomation in $GL(V)$ such that*

$$g \cdot v = \rho(g)v, \quad \forall g \in G, v \in V,$$

then the realization is called a **representation**.

Definition. *If G is a Lie group, an action of G on a finite-dimensional vector space V is said to be* **linear** *if $\forall g \in G$, the map from V to itself given by $v \mapsto g \cdot v$ is linear.*

- Smooth linear actions correspond precisely to representations.
- The image of a representation $\rho : G \to \mathrm{GL}(V)$ is a Lie subgroup of $\mathrm{GL}(V)$.
- If the representation $\rho : G \to \mathrm{GL}(V)$ is faithful, it gives a Lie group isomorphism between G and $\rho(G) \subset \mathrm{GL}(V)$.

— By choosing a basis of V, we obtain a Lie group isomorphism $\mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$, so a Lie group admits a **faithful** representation iff it is isomorphic to a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$ for some n.

Examples (Lie Group Representations).

(a) If G is any Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, the inclusion map

$$G \hookrightarrow \mathrm{GL}(n, \mathbb{R}) = \mathrm{GL}(\mathbb{R}^n)$$

is a faithful representation, called the **defining representation** of G.
The defining representation of a subgroup of $\mathrm{GL}(n, \mathbb{C})$ is defined similarly.

(b) The inclusion map

$$\mathbb{S}^1 \hookrightarrow \mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$$

is a faithful representation of the circle group.

— More generally, the map $\mathbb{T}^n \to \mathrm{GL}(n, \mathbb{C})$ given by

$$\rho(z^1, \cdots, z^n) = \begin{pmatrix} z^1 & 0 & \cdots & 0 \\ 0 & z^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z^n \end{pmatrix}$$

is a faithful representation of $\mathbb{T}^n$.

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(c) Let $\sigma : \mathbb{R}^n \to \mathrm{GL}(n+1, \mathbb{R})$ be a map that sends $x \in \mathbb{R}^n$ to the matrix $\sigma(x)$ defined in block form by

$$\sigma(x) = \begin{pmatrix} I_n & x \\ 0 & 1, \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix and x is regareded as $n \times 1$ column matrix.

— A straightforward and σ is a faithful representation of the additive Lie group $\mathbb{R}^n$.

(d) Another faithful representation of $\mathbb{R}^n$ is the map $\mathbb{R}^n \to \mathrm{GL}(n, \mathbb{R})$ that sends $x = (x^1, \cdots, x^n) \in \mathbb{R}^n$ to the diagonal matrix whose diagonal entries are $(e^{x^1}, \cdots, e^{x^n})$.

(e) Yet another representation of $\mathbb{R}^n$ is the map $\mathbb{R}^n \to \mathrm{GL}(n, \mathbb{C})$ sending x to the diaginal matrix with diagonal entries $(e^{2\pi i x^1}, \cdots, e^{2\pi i x^n})$.

— This one is **not faithful**, because its kernel is the subgroup $\mathbb{Z}^n \subset \mathbb{R}^n$.

Definition. *If G and H are Lie groups, a* **Lie group homomorphism** *from G to H is smooth map $F : G \to H$ that is also a group homomorphiem.*

- *It is called a* **Lie group isomorphism** *if it is also a diffeomorphism, (which implies that it has an inverse that is also a Lie group homomorphism). In this case we say that G and H are* **isomorphic Lie groups**.

Induced Lie Algebra Homomorphisms

- The importance of the Lie algebra of a Lie group stems, in a large part, from the fact that each Lie group homomorphism induces a Lie algebra homomorphism.

Theorem 1. *Let G and H be Lie groups, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras.*
Suppose $F : G \to H$ is a Lie group homomorphism.
For every $X \in \mathfrak{g}$, there is a unique vector field in $\mathfrak{h}$ that is F-related to X.
With this vector field denoted by F_*X, the map $F_* : \mathfrak{g} \to \mathfrak{h}$ so defined is a Lie algebra homomorphism.

Proof. If there is any vector field $Y \in \mathfrak{h}$ that is F-related to X, it must satisfy $Y_e = F_*X_e$, and thus it must be uniquely determined by

$$Y = \widetilde{F_*X_e}.$$

(i) **To show that Y is F-related to X**, note that the fact that F is a homomorphism implies

$$\begin{aligned} F(gg') = F(g)F(g') \Longrightarrow & F(L_g g') = L_{F(g)}F(g') \\ \Longrightarrow & F \circ L_g = L_{F(g)} \circ F \\ \Longrightarrow & F_* \circ (L_g)_* = (L_{F(g)})_* F_*. \end{aligned}$$

Thus

$$F_*X_g = F_*(L_g)_*X_e = (L_{F(g)})_*F_*X_e = (L_{F(g)})_*Y_e = Y_{F(g)}.$$

This says precisely that X and Y are F-related.

(ii) $\forall X \in \mathfrak{g}$, let F_*X denote the unique vector field in $\mathfrak{h}$ that is F-related to X.
It then follows immediately from the naturality of Lie brackets that

$$F_*[X, Y] = [F_*X, F_*Y],$$

so F_* is a Lie algebra homomorphism. □

Definition. *The map $F_* : \mathfrak{g} \to \mathfrak{h}$ whose existence is asserted in this theorem will be called the* **induced Lie algebra homomorphism**.

Proposition 2. *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. If H is another Lie group and $\mathfrak{h}$ is its Lie algebra, for any Lie group homomorphism $F : G \to H$, the following diagram commutes:*

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "F_*"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\
G \arrow[r, "F"'] & H.
\end{tikzcd}$$

Proof. We need to show that $\exp(F_* X) = F(\exp X)$ for every $X \in \mathfrak{g}$.
In fact, we will show that for all $t \in \mathbb{R}$,

$$\exp(tF_* X) = F(\exp tX).$$

The left hand side is, by (b), the one-parameter subgroup generated by $F_* X$. Thus if we put $\sigma(t) = F(\exp tX)$, it suffices to show that $\sigma : \mathbb{R} \to H$ is a group homomorphism satisfying $\sigma'(0) = F_* X$. Indeed, we compute

$$\sigma'(0) = \frac{d}{dt}\bigg|_{t=0} F(\exp tX) = F_*(\frac{d}{dt}\bigg|_{t=0} \exp tX) = F_* X$$

and

$$\begin{aligned}
\sigma(s+t) =& F(\exp(s+t)X) \\
=& F(\exp sX \exp tX) \quad \text{by (c)} \\
=& F(\exp sX) F(\exp tX) \quad \text{since } F \text{ is a homomorphism} \\
=& \sigma(s)\sigma(t). \qquad \square
\end{aligned}$$

Proposition 3. *The flow θ of a left-invariant vector field X is given by*

$$\theta_t = R_{\exp tX} \quad \textit{(right multiplication by } \exp tX\textit{)}.$$

Proof. Let θ_t be the one-parameter group generated by X. We have

$$R_{\theta_t(e)}(g) = g(\theta_t(e)) = L_g(\theta_t(e)) = \theta_t(L_g(e)) = \theta_t(g);$$

that is,

$$R_{\exp tX}(g) = \theta_t(g). \qquad \square$$

Adjoint Representation

- Consider first the map of G into itself given by

$$I_g : h \mapsto ghg^{-1}.$$

  This is the **group adjoint realization** of G consisting of left-translation by g and right-translation by g^{-1}.

⊙ If G is abelian, then I_g is the identity map $h \mapsto h$, $\forall g \in G$.

- Notice that each I_g maps the identity e into itself, so that every curve through e is mapped into a (possibly different) curve through e.

– Thus I_g induces a map $(I_g)_*$ mapping any vector in T_e to another one in T_e.

– This map is called $\mathrm{Ad}(g)$, **the adjoint transformation of T_e induced by g.**

$$\mathrm{Ad}(g)(X_e) = (I_g)_*(X_e).$$

- Setting $\sigma(t) = I_g(\exp tX)$, we obtain

$$\begin{aligned}\sigma(s+t) =& I_g(\exp(s+t)X)\\ =& I_g(\exp(sX)\exp(tX))\\ =& I_g(\exp(sX)I_g(\exp tX) \quad \text{since } I_g \text{ is a group homomorphism}\\ =& \sigma(s)\sigma(t).\end{aligned}$$

  Hence, $I_g(\exp(tX))$ is a one-parameter subgroup, with

$$\sigma'(0) = \frac{d}{dt}\bigg|_{t=0} I_g(\exp tX) = (I_g)_*(\frac{d}{dt}\bigg|_{t=0} \exp tX) = (I_g)_* X.$$

  Hence

$$I_g(\exp tX) = \exp(t(I_g)_* X) = \exp(t\mathrm{Ad}(g)(X)).$$

  In other words, the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}(g)"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "I_g"'] & G. \end{tikzcd}$$

Proposition. *$Ad\,(g)(X) = (I_g)_*(X) = (R_{g^{-1}})_*(L_g)_* X = (R_{g^{-1}})_* X$.*

- Now if g itself is a member of a one-parameter subgroup $g(s) = \exp(sY)$, there should be a natural expression for $\mathrm{Ad}(g)X$ in terms of Y. Indeed, we have

Theorem 4.

$$\frac{d}{dt}\bigg|_{t=0} Ad(\exp tY)X = (\mathcal{L}_Y X)_e = [Y, X]\bigg|_e.$$

$$(Ad(\exp tY))_* X = [Y, X].$$

Proof. Let $X \in \mathfrak{g}$ be arbitrary. Because $t \mapsto \exp tX$ is a smooth curve in G whose tangent vector at $t = 0$ is X, we can compute $\operatorname{Ad}(g)(X)$ by

$$\operatorname{Ad}(g)X = \left.\frac{d}{dt}\right|_{t=0} I_g(\exp tX).$$

$$\begin{aligned}\therefore \operatorname{Ad}(\exp tY)X =& \left.\frac{d}{dt}\right|_{t=0} I_{\exp tY}(\exp tX)\\ =&(I_{\exp tX})_*\left(\left.\frac{d}{dt}\right|_{t=0} \exp tX\right) = (I_{\exp tY})_* X.\end{aligned}$$

Using the fact that $(I_g)_* = (R_{g^{-1}})_* \circ (L_g)_*$, its value at $e \in G$ can be computed as

$$\begin{aligned}(\operatorname{Ad}(\exp tY)X)_e =&(R_{\exp(-tY)})_*(L_{\exp tY})_* X_e\\ =&(R_{\exp(-tY)})_* X_{\exp tY}.\end{aligned} \tag{1}$$

⊙ Recall from Proposition 3 that the flow of Y is given by $\theta_t(g) = R_{\exp tY}(g)$. Therefore, (1) can be rewritten as

$$(\operatorname{Ad}(\exp tY)X)_e = (\theta_{-t})_* X_{\theta_t(e)}.$$

Taking the derivative with respect to t and setting $t = 0$, we obtain

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Ad}(\exp tY)X = \left.\frac{d}{dt}\right|_{t=0} (\theta_{-t})_* X_{\theta_t(e)} = (\mathcal{L}_Y X)_e = [Y,X]\Big|_e . \quad \square$$

Theorem 4*. $\frac{\partial}{\partial t}\frac{\partial}{\partial s}((\exp tY)(\exp sX)(\exp -tY))\Big|_{t=0,s=0} = [Y,X]\Big|_e .$

Proof. Let θ_t be the one-parameter group of diffeomorphisms generated by Y. Then

$$[X,Y]\Big|_e = (\mathcal{L}_Y X)_e = \left.\frac{d}{dt}\right|_{t=0} (\theta_{-t})_* X_{\theta_t(e)}.$$

Now the integral curve of X though $\theta_t(e) = \exp(tY)$ is equal to $\exp(tY)\exp(sX)$ by the left invariance of X. Then

$$[X,Y]\Big|_e = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\theta_{-t}(\exp(tY)\exp(sX))\Big|_{t=0,s=0}.$$

By left invariance of the integral curves of X we then have

$$\theta_{-t}(g) = g\theta_{-t} = g\exp(-tX), \quad \forall g \in G.$$

Hence

$$[Y,X]\Big|_e = \frac{\partial}{\partial t}\frac{\partial}{\partial s}((\exp tY)(\exp sX)(\exp -tX))\Big|_{t=0,s=0}. \quad \square$$