Homogeneous Spaces

- One of the most interesting kinds of group action is that in which a group acts transitively.

Definition. *A smooth manifold endowed with a transitive smooth action by a Lie group G is called a* **homogeneous G-space**.

- In most examples of homogeneous spaces, the group action preserves some property of the manifold (such as distances in some metric, or a class of curves such as straight lines in the plane); then the fact that the action is **transitive** means that the manifold "looks the same" everywhere from the point of view of this property.
- Often, homogeneous spaces are models for various kinds of geometric structures, and as such they play a central role in many area of differential geometry.

Examples (Homogeneous Spaces).

(a) The natural action of $O(n)$ on $\mathbb{S}^{n-1}$ is transitive.
— So is the natural action of $SO(n)$ on $\mathbb{S}^{n-1}$ when $n \geq 2$.
— Thus, for $n \geq 2$, $\mathbb{S}^{n-1}$ is a homogeneous space of either $O(n)$ or $SO(n)$.

(b) Let $E(n)$ denote the subgroup of $\mathrm{GL}\,(n+1, \mathbb{R})$ consisting of matrices of the form

$$\left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} : A \in O(n), b \in \mathbb{R}^n \right\},$$

where b is considered as an $n \times 1$ column matrix.
— It is straightforward to check that $E(n)$ is an embedded Lie subgroup.
— If $S \subset \mathbb{R}^{n+1}$ denotes the affine subspace defined by $x^{n+1} = 1$, then a simple computation shows that $E(n)$ takes S to itself.
— If we identify S with $\mathbb{R}^n$, in which the matrix $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$ sends x to $Ax + b$.
— It is not hard to prove that these are precisely the diffeomorphisms of $\mathbb{R}^n$ that preserves the Euclidean distance function.
– For this reason, $E(n)$ is called the **Euclidean group**.
— Because any point in $\mathbb{R}^n$ can be taken to any other by a translation, $E(n)$ acts transitively on $\mathbb{R}^n$, so $E(n)$ acts transitively on $\mathbb{R}^n$, so $\mathbb{R}^n$ is a homogeneous $E(n)$-space.

(c) The group $\mathrm{SL}\,(2, \mathbb{R})$ acts smoothly and transitively on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

The resulting complex-analytic transformation of $\mathbb{H}$ are called **Mobiüs trnsformations**.

(d) The natural action of $\mathrm{GL}\,(n, \mathbb{C})$ on $\mathbb{C}^n$ restricts to natural actions of both $U(n)$ and $SU(n)$ on $\mathbb{S}^{2n-1}$, thought of as the set of unit vectors in $\mathbb{C}^n$. The natural actions of $U(n)$ and $SU(n)$ on $\mathbb{S}^{n-1}$ are smooth and transitive.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

- Next we describe a very general construction that can be used to generate homogeneous spaces as quotients of Lie groups by closed Lie subgroups.

Definition. *Let G be a Lie group and $H \subset G$ be a Lie subgroup. For each $g \in G$, the* **left coset** *of g modulo H is the set*

$$gH = \{gh : h \in H\}.$$

The set of left costs modulo H is denoted by G/H;
with the quotient topology determined by the natural map $\pi : G \to G/H$ sending each element $g \in G$ to its coset, it is called the **left coset space of G modulo H**.

- Two elements g_1, $g_2 \in G$ are in the same coset modulo H iff $g_2^{-1} g_2 \in H$; in this case we write $g_1 \equiv g_2 (\mathrm{mod} H)$.

Theorem 1 (Homogeneous Space Construction Theorem). *Let G be a Lie group and H be a closed Lie subgroup of G. The left coset space G/H has a unique smooth manifold structure such that the quotient map*

$$\pi : G \to G/H$$

is a smooth submersion. The left action of G on G/H given by

$$g_1 \cdot (g_2 H) = (g_1 g_2) H \tag{1}$$

turns G/H into a homogeneous G-space.

Proof. If we let H act on G by right translation, then
g_1, $g_2 \in G$ are in the same H-orbit$\Leftrightarrow g_1 h = g_2$ for some $h \in H$,
$\Leftrightarrow g_1$ and g_2 are in the same coset modulo H.

— In other words, the orbit space determined by the **right** action of H on G is precisely the **left** coset space G/H.

⊙ H acts smoothly and freely on G.

⊙ To see that the action is **proper**, suppose $\{g_i\}$ is a convergent sequence in G and $\{h_i\}$ is a sequence in H such that $\{g_i h_i\}$ converges.

— By **continuity**, $h_i = g_i^{-1}(g_i h_i)$ converges to a point in G,
and since H is closed in G it follows that $\{h_i\}$ converges in H.

⊙ The **quotient manifold theorem** now implies that G/H has a unique smooth manifold structure such that the quotient map $\pi : G \to G/H$ is a submersion.

⊙ Since a product of submersions is a submersion, it follows that

$$\mathrm{Id}_G \times \pi : G \times G \to G \times G/H$$

is also a submersion. Consider the following diagram:

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ {\scriptstyle \mathrm{Id}_G \times \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ G \times G/H & \xrightarrow[\theta]{} & G/H, \end{array}$$

where m is a group multiplication and θ is the action of G on G/H given by (1).

— It is straightforward to check that $\pi \circ m$ is constant on the fibers of $\mathrm{Id}_G \times \pi$, and therefore θ is **well-defined** and **smooth**.

⊙ Finally, given any two points $g_1 H$, $g_2 H \in G/H$, the element $g_2 g_1^{-1} \in G$ satisfies $(g_2 g_1^{-1}) \cdot g_1 H = g_2 H$, so the action is **transitive**. □

- The homogeneous spaces constructed in this theorem turn out to be of central importance because, as the next theorem shows, **every homogeneous space is equivalent to one of this type**.

— First we note that the isotropy group of any smooth Lie group action is a closed Lie subgroup.

Lemma 2. *If M is a smooth G-space, then for each $p \in M$, the isotropy group G_p is a closed, embedded Lie subgroup of G.*

Proof. For each $p \in M$, $G_p = (\theta^{(p)})^{-1}(p)$, where $\theta^{(p)} : G \to M$ is the orbit map

$$\theta^{(p)}(g) = g \cdot p.$$

⊙ **Claim: $\theta^{(p)}$ has constant rank**. Indeed, since

$$\theta^{(p)}(g'g) = g' \cdot \theta^{(p)}(g),$$

the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\theta^{(p)}} & M \\ {\scriptstyle L_{g'}}\big\downarrow & & \big\downarrow{\scriptstyle \theta^{(p)}(g')} \\ G & \xrightarrow[\theta^{(p)}]{} & M \end{array} \qquad\qquad \begin{array}{ccc} T_gG & \xrightarrow{(\theta^{(p)})_*} & T_{\theta^{(p)}(g)}M \\ {\scriptstyle (L_{g'})_*}\big\downarrow & & \big\downarrow{\scriptstyle (\theta^{(p)}(g'))_*} \\ T_{g'g}G & \xrightarrow[(\theta^{(p)})_*]{} & T_{\theta^p(g'g)}M, \end{array}$$

Because the vertical linear maps in the second diagram are isomorphisms, the horizontal lines have the same rank. In other words, the rank of $(\theta^{(p)})_*$ at an arbitrary point p is the same as its rank at $g \cdot p$, so $\theta^{(p)}$ has constant rank.

— Hence $G_p = (\theta^{(p)})^{-1}(p)$ is an embedded submanifold of G, and therefore is a closed Lie subgroup. □

Theorem 3 (Homogeneors Space Characterization Theorem). *Let M be a homogeneous G-space, and let p be any point of M. Then the map*

$$F : G/G_p \to M$$

defined by $F(gG_p) = g \cdot p$ is a diffeomorphism.

Proof. To see that F **is well-defined**, assume that $g_1G_p = g_2G_p$, which means that $g_2^{-1}g_1 \in G_p$. Writing $g_2^{-1}g_1 = h$, we see that

$$F(g_2G_p) = g_2 \cdot p = g_1h \cdot p = g_1 \cdot p = F(g_1G_p).$$

- It is **smooth** because it is obtained from the orbit map $\theta^{(p)} : G \to M$ by passing to the quotient.
- Next we show that F is **bijective**.
  (1) Given any point $q \in M$ there is a group element $g \in G$ such that
  $F(gG_p) = g \cdot p = q$ by transitivity.
  (2) On the other hand, if $F(g_1G_p) = F(g_2G_p) \Rightarrow g_1 \cdot p = g_2 \cdot p$.
  $\Rightarrow g_1^{-1}g_2 \cdot p = p. \Rightarrow g_1^{-1}g_2 \in G_p. \Rightarrow g_1G_p = g_2G_p.$

- Also, F has constant rank, since the relation

$$F(g'gG_p) = (g'g) \cdot p = g' \cdot F(gG_p)$$

yields that following diagrams commute:

$$\begin{array}{ccc} G/G_p & \xrightarrow{F} & M \\ L_{g'}\big\downarrow & & \big\downarrow \theta^{(p)}(g') \\ G/G_p & \xrightarrow[F]{} & M \end{array} \qquad \begin{array}{ccc} T_gG & \xrightarrow{F_*} & T_{\theta^{(p)}(g)}M \\ (L_{g'})_*\big\downarrow & & \big\downarrow (\theta^{(p)}(g'))_* \\ T_{g'g}G & \xrightarrow[F_*]{} & T_{\theta^p(g'g)}M, \end{array}$$

Because the vertical linear maps in the second diagram are isomorphisms, the horizontal lines have the same rank. In other words, the rank of F_* at an arbitrary point gG_p is the same as its rank at $gg'G_p$, so F has constant rank.
- Because F has constant rank and is bijective, it is a diffeomorphism. $\square$

Examples (Homogeneous Spaces).

(a) Consider again the natural action of $\mathrm{O}(n)$ on $\mathbb{S}^{n-1}$.
If we choose our base point in $\mathbb{S}^{n-1}$ to be the "north pole" $N = (0, \cdots, 0, 1)$, then the isotropy group is $\mathrm{O}(n-1)$, thought of an orthogonal transformations of $\mathbb{R}^n$ that fix the last variable.
Thus $\mathbb{S}^{n-1}$ is diffeomorphic to the quotient manifold $O(n)/O(n-1)$.

⊙ For the action of $\mathrm{SO}(n)$ on $\mathbb{S}^{n-1}$, the isotropy group is $\mathrm{SO}(n-1)$, so $\mathbb{S}^{n-1}$ is also diffeomorphic to $SO(n)/SO(n-1)$.

(b) The Euclidean group $E(n)$ acts smoothly and transitively on $\mathbb{R}^n$,
and the isotropy group of the origin is the subgroup $\mathrm{O}(n) \subset E(n)$ (identified with the $(n+1) \times (n+1)$ matrices of the form $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ with $A \in O(n)$).
Hence $\mathbb{R}^n$ is diffeomorphic to $E(n)/O(n)$.

(c) Consider the transitive action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{H}$ by Möbius transformations.
Direct computatoion shows that the isotropy group of the point $i \in \mathbb{H}$ consists of matrices of the form $\begin{pmatrix} -a & b \\ -b & a \end{pmatrix}$ with $a^2 + b^2 = 1$.
This subgroup is exactly $\mathrm{SO}(2) \subset \mathrm{SL}(2, \mathbb{R})$, so the characteristic theorem gives rise to a diffeomorphism $H \approx SL(2, \mathbb{R})/SO(2)$.

(d) $\mathbb{S}^{n-1} \approx U(n)/U(n-1) \approx SU(n)/SU(n-1)$.