The Model Spaces of Riemannian Geometry

Spheres

Conformal Equivalence between $\mathbb{R}^n$ and $\mathbb{S}^n_R \setminus \{N\}$

- A **conformal equivalence** between $\mathbb{R}^n$ and $\mathbb{S}^n_R \subset \mathbb{R}^{n+1}$ minus a point is provided by **stereographic projection** from the north pole:

$$\sigma : \mathbb{S}^n_R \setminus \{N\} \to \mathbb{R}^n$$

$$P = (\xi^1, \cdots, \xi^n, \tau) \mapsto u = (u^1, \cdots, u^n)$$

where $U = (u^1, \cdots, u^n, 0) = (u, 0)$ is the point where the line through N to P intersects the hyperplane $\{\tau = 0\}$ in $\mathbb{R}^{n+1}$.

- Thus U is characterized by the fact that $\overline{NU} = \lambda \overline{NP}$ for some nonzero scalar λ. Writing $N = (0, R)$, $U = (u, 0)$, and $P = (\xi, \tau) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, this leads to the system of equations

$$\begin{aligned} u^i &= \lambda \xi^i, \\ -R &= \lambda (R - \tau). \end{aligned} \tag{1}$$

Solving the second equation for λ and plugging it into the first equation, we obtain the formula for the stereographic projection

$$\sigma(\xi, \tau) = u = \frac{R\xi}{R - \tau}. \tag{2}$$

- To compute the inverse of σ, we solve the two equations of (1) for τ and ξ^i and obtain

$$\xi^i = \frac{u^i}{\lambda}, \quad \tau = R\frac{\lambda - 1}{\lambda}. \tag{3}$$

The point $P = \sigma^{-1}(u)$ is charaterized by these equations and the fact that P is on the sphere $\mathbb{S}^n_R$. Substituting (3) into $|\xi|^2 + \tau^2 = R^2$ gives

$$\frac{|u|^2}{\lambda^2} + R^2 \frac{(\lambda - 1)^2}{\lambda^2} = R^2,$$

from which we obtain

$$\lambda = \frac{|u|^2 + R^2}{2R^2}.$$

Inserting this back into (3) gives the formula

$$\sigma^{-1}(u) = (\xi, \tau) = \left(\frac{2R^2 u}{|u|^2 + R^2}, R\frac{|u|^2 - R^2}{|u|^2 + R^2} \right),$$

which maps $\mathbb{R}^n$ back to $\mathbb{S}^n_R \setminus \{N\}$ and shows that σ is a diffeomorphism.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Lemma. *Stereographic projection is a conformal equivalence between* $\mathbb{S}^n_R \setminus \{N\}$ *and* $\mathbb{R}^n$.

Proof. The inverse map σ^{-1} is a local parametrization, so we will use it to compute the pullback metric. Consider an arbitrary point $q \in \mathbb{R}^n$ and a vector $V \in T_q\mathbb{R}^n$, and compute

$$(\sigma^{-1})^*\overline{g}(V,V) = \overline{g}(\sigma_*^{-1}V, \sigma_*^{-1}V)(= \mathring{g}_R(\sigma_*^{-1}V, \sigma_*^{-1}V))$$

where $\overline{g}_{\mathbb{R}^{n+1}}$ and $\overline{g}_{\mathbb{R}^n}$ denotes the Euclidean metric on $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, respectively. We **claim:**

$$(\sigma^{-1})^*\overline{g}_{\mathbb{R}^{n+1}} = \frac{4R^4}{(|u|^2+R^2)^2}\overline{g}_{\mathbb{R}^n},$$

— Writing $V = V^i\partial_i$ and $\sigma^{-1}(u) = (\xi(u), \tau(u))$, we have

$$\sigma_*^{-1}V = V^i\frac{\partial \xi^j}{\partial u^i}\frac{\partial}{\partial \xi^j} + V^i\frac{\partial \tau}{\partial u^i}\frac{\partial}{\partial \tau},$$

in which

$$V\xi^i = V\left(\frac{2R^2u^j}{|u|^2+R^2}\right) = \frac{2R^2V^j}{|u|^2+R^2} - \frac{4R^2u^j\langle V,u\rangle}{(|u|^2+R^2)^2},$$

$$V\tau = V\left(R\frac{|u|^2-R^2}{|u|^2+R^2}\right) = \frac{2R\langle V,u\rangle}{|u|^2+R^2} - \frac{2R(|u|^2-R^2)\langle V,u\rangle}{(|u|^2+R^2)^2} = \frac{4R^3\langle V,u\rangle}{(|u|^2+R^2)^2},$$

where we have used the notation $V(|u|^2) = 2\sum_k V^k u^k = 2\langle V,u\rangle$. Therefore,

$$\begin{aligned}\overline{g}(\sigma_*^{-1}V, \sigma_*^{-1}V) =& \sum_{j=1}^n (V\xi^j)^2 + (V\tau)^2 \\ =& \frac{4R^4|V|^2}{(|u|^2+R^2)^2} - \frac{16R^4\langle V,u\rangle^2}{(|u|^2+R^2)^3} + \frac{16R^4|u|^2\langle V,u\rangle^2}{(|u|^2+R^2)^4} + \frac{16R^6\langle V,u\rangle^2}{(|u|^2+R^2)^4} \\ =& \frac{4R^4|V|^2}{(|u|^2+R^2)^2}. \qquad \square\end{aligned}$$

Corollary. *The sphere is* **locally conformally flat**; *i.e. each point* $p \in \mathbb{S}^n_R$ *has a nbhd that is conformally equivalent to an open set in* $\mathbb{R}^n$.

Proof. Stereographic projection gives such an equivalence for a nbhd of any point except the north pole.
Apply a suitable rotation and then stereographic projection (or a stereographic projection from the south pole), we get auch an equivalence for a nbhd of the north pole. $\square$

Hyperbolic Spaces

Proposition 3. *For any fixed $R > 0$, the following Riemannian manifolds are all mutually isometric.*

(1) **Hyperbolic Model** $\mathbb{H}^n_R$ *is the "upper sheet" ($\{\tau > 0\}$) of the two-sheeted hyperboloid in $\mathbb{R}^{n+1}$ defined in coordinates $(\xi^1, \cdots, \xi^n, \tau)$ by the equation $\tau^2 - |\xi|^2 = R^2$, with the metric*

$$h^1_R = \imath^* m,$$

where $\imath : \mathbb{H}^{n+1}_R \hookrightarrow \mathbb{R}^{n+1}$ is inclusion, and m is the Minkowski metric on $\mathbb{R}^{n+1}$.

(2) **Poincaré Model** $\mathbb{B}^n_R$ *is the ball of radius R in $\mathbb{R}^n$, with the metric given in coordinates $(u^1, \cdots, u^n)$ by*

$$h^2_R = 4R^4 \frac{(du^1)^2 + \cdots + (du^n)^2}{(R^2 - |u|^2)^2}.$$

(3) **Poincaré Half-space Model** $\mathbb{U}^n_R$ *is the upper half-space in $\mathbb{R}^n$ defined in coordinates $(x^1, \cdots, x^n, y)$ by $\{y > 0\}$, with the metric*

$$h^3_R = R^2 \frac{(dx^1)^2 + \cdots + (dx^{n-1})^2 + dy^2}{y^2}.$$

Proof. **(I) Claim: (1)⇔(2).**
We begin by giving a geometric construction of a diffeomorphism

$$\pi : \mathbb{H}^n_R \to \mathbb{B}^n_R$$

from the hyperboloid to the ball, which we call **hyperbolic stereographic projection**, which turns out to be an isometry between the two metrics given in (1) and (2).

— Let $S \in \mathbb{R}^{n+1}$ be the point $S = (0, \cdots, 0, -R)$. For any $P \in \mathbb{H}^n_R \subset \mathbb{R}^{n+1}$, set

$$\pi(P) = u \in \mathbb{B}^n_R,$$

$$P = (\xi^1, \cdots, \xi^n, \tau) \mapsto u = (u^1, \cdots, u^n)$$

where $U = (u^1, \cdots, u^n, 0) = (u, 0) \in \mathbb{R}^{n+1}$ is the point where the line through S to P intersects the hyperplane $\{\tau = 0\}$ in $\mathbb{R}^{n+1}$.

— Thus U is characterized by the fact that $\overline{SU} = \lambda \overline{SP}$ for some nonzero scalar λ,

$$u^i = \lambda \xi^i,$$
$$R = \lambda(R + \tau).$$

Solving these equations, we obtain

$$\pi(\xi, \tau) = u = \frac{R\xi}{R + \tau}. \tag{2*}$$

and its inverse map

$$\sigma^{-1}(u) = (\xi, \tau) = \left(\frac{2R^2 u}{R^2 - |u|^2}, R\frac{R^2 + |u|^2}{R^2 - |u|^2}\right).$$

— Consider an arbitrary point $q \in \mathbb{R}^n$ and a vector $V \in T_q\mathbb{R}^n$, and compute

$$(\pi^{-1})^* h_R^1(V, V) = h_R^1(\pi_*^{-1}V, \pi_*^{-1}V) = m(\pi_*^{-1}V, \pi *^{-1} V).$$

— Writing $V = V^i\partial_i$ and $\sigma^{-1}(u) = (\xi(u), \tau(u))$, we have

$$\pi_*^{-1}V = V^i \frac{\partial \xi^j}{\partial u^i}\frac{\partial}{\partial \xi^j} + V^i\frac{\partial \tau}{\partial u^i}\frac{\partial}{\partial \tau},$$

in which

$$V\xi^i = V\left(\frac{2R^2u^j}{R^2 - |u|^2}\right) = \frac{2R^2V^j}{R^2 - |u|^2} + \frac{4R^2u^j\langle V, u\rangle}{(R^2 - |u|^2)^2},$$

$$V\tau = V\left(R\frac{R^2 + |u|^2}{R^2 - |u|^2}\right) = \frac{4R^3\langle V, u\rangle}{(R^2 - |u|^2)^2}$$

Therefore,

$$\begin{aligned} m(\pi_*^{-1}V, \pi_*^{-1}V) =& \sum_{j=1}^{n}(V\xi^j)^2 - (V\tau)^2 \\ =& \frac{4R^4|V|^2}{(R^2 - |u|^2)^2} \\ =& h_R^2(V, V). \qquad \square \end{aligned}$$

(II) Claim: (2)⇔(3).

Next we consider the Poincaré half-space model, by constructing an explicit diffeomorphism

$$\kappa : \mathbb{B}_R^n \to \mathbb{U}_R^n.$$

In this case it is more convenient to write the coordinates on the ball as

$$(u^1, \cdots, u^{n-1}, v) = (u, v).$$

(i) In the 2-dimensional case, κ is easy to write down in complex natation

$$w = u + iv, \quad \text{and} \quad z = x + iy.$$

It is a variant of the classical Cayley transform:

$$\kappa(w) = z = -iR\frac{w + iR}{w - iR},$$

which is a complex-analytic diffeomorphism taking $\mathbb{B}_R^2$ to $\mathbb{U}_R^2$.

— Seperating z into real and imaginary parts, this can be written in real terms as

$$\kappa(u,v) = (x,y) = \left(\frac{2R^2 u}{|u|^2 + (v-R)^2}, R\frac{R^2 - |u|^2 - v^2}{|u|^2 + (v-R)^2} \right).$$

(ii) The same formula makes sense **in any dimension**, and obviously maps the ball $\{|u|^2 + v^2 < R^2\}$ into the upper half-space.

— It is straightforward to check that its inverse is

$$\kappa^{-1}(x,y) = (u,v) = \left(\frac{2R^2 x}{|x|^2 + (y+R)^2}, R\frac{|x|^2 + |y|^2 - R^2}{|x|^2 + (y+R)^2} \right).$$

Hence κ is a diffeomorphism, called **the generalized Cayley transform**.

— The verification $\kappa^* h_R^3 = h_R^2$ is a long calculation. □

- We often use the notation $\mathbb{H}_R^n$ to refer to any one of the manifolds of Proposition 3, and h_R to refer to the corresponding metric, using whichever model is most convenient for the application we have in mind.

Corollary. *The hyperbolic metric is locally conformally flat.*

Proof. The form of the metric in either the ball model or the half-space model makes it clear that the identity map gives a **global** conformal equivalence with an open subset of Euclidean space. □