The Conjugate and Cut Loci

Let M be a Riemannian manifold, and $\mathcal{E}$ is the dmain of the exponential map.

Definition. *(1) Let $p \in M$. The* **conjugate locus of** p **in** T_pM *is the subset of $T_pM \cap \mathcal{E}$ consisting of all critical points of* $\exp_p$.

(2) By the **conjugate locus of** p **in** M, *we mean the image of the conjugate locus of p in T_pM under the exponential map* $\exp_p$.

Definition. *Given $p \in M$, $\xi \in B_1(0_p)$, we define $c(\xi)$ the* **distance to the cut point of** p **along** γ_ξ **by**

$$c(\xi) = \sup\{t > 0 : t\xi \in \mathcal{E}, d(p, \gamma_\xi(t)) = t\}.$$

If $c(\xi) < \infty$, $\gamma_\xi(c(\xi))$ is called the cut point of p along γ.

Definition. *For every $p \in M$, define the* **cut locus of** p **in** T_pM $\mathbf{C}(p)$, *by*

$$\mathbf{C}(p) = \{c(\xi)\xi : c(\xi) < \infty,\ \xi \in \partial B_1(0_p)\} \cap \mathcal{E},$$

and the **cut locus of** p **in** M, $C(p)$, *by*

$$C(p) = \exp \mathbf{C}(p).$$

Also, we set

$$\mathbf{D}_p = \{t\xi : 0 < t < c(\xi),\ \xi \in \partial B_1(0_p)\},$$
$$D_p = \exp \mathbf{D}_p.$$

Lemma. *If $d(p, \gamma_\xi(t_1)) = t_1$ for some given $t_1 > 0$, then $d(p, \gamma_\xi(t)) = t$ for all $t \in [0, t_1]$.*

Proof. If there exists $T \in [0, t_1]$ such that $d(p, \gamma_\xi(T)) < T$, then the triangle inequality implies

$$\begin{aligned} d(p, \gamma_\xi(t_1)) \le & d(p, \gamma_\xi(T)) + d(\gamma_\xi(T), \gamma_\xi(t_1)) \\ < & T + (t_1 - T) = t_1. \quad \square \end{aligned}$$

Proposition 1. *Geodesics minimizes distance from p to $\gamma_\xi(t)$ for all $t \in [0, c(\xi)]$, and fails to minimize distance for all $t > c(\xi)$.*
If $c(\xi) < \infty$ and $c(\xi)\xi \in \mathcal{E}$, then γ_ξ minimizes distance between ξ and $\gamma_\xi(c(\xi))$.

Proposition 2. *If $t < c(\xi)$, then γ_ξ is the* **unique** *minimizing geodesic from p to $\gamma_\xi(t)$.*

Proof. If not, then there exists another $\eta \in \partial B_1(0_p)$ for which $\gamma_\eta(t) = \gamma_\xi(t)$.
But then, for any $T \in (t, c(\xi))$, one could travel along γ_η from p to $\gamma_\xi(t)$ followed by traveling along γ_ξ from $\gamma_\xi(t)$ to $\gamma_\xi(T)$.
Then, one would have a minimizing broken geodesic from p to $\gamma_\xi(T)$, a contradiction. $\square$

- **A more detailed description of** $c(\xi)$:

— Certainly, if p has a conjugate point $\gamma_\xi(T)$ along γ_ξ, then $c(\xi) \leq T$.
— Thus, one possibility for $c(\xi)$ is that the distance along γ_ξ to the first conjugate point of p along γ_ξ.

Whar are the other possibilities?

When M is complete, there is only one other possibility is that there are at least two distinct minimizing geodesics from p to $\gamma_\xi(c(\xi))$, as stated in (2) in the following:

Proposition 3. *Let M be complete. Suppose that $\gamma(t_0)$ is the cut point of $p = \gamma(0)$ along γ. Then*

(1) *either $\gamma(t_0)$ is the first conjugate point of $\gamma(0)$ along γ,*
(2) *or there exists a geodesic $\sigma \neq \gamma$ from p to $\gamma(t_0)$ such that $L(\sigma) = L(\gamma)$.*

Conversely, if (1) or (2) is satisfied, then there exists $\tilde{t} \in (0, t_0]$ such that $\gamma(\tilde{t})$ is the cut point of p along γ.

Example. On the cylinder $\mathbb{S}^1 \times \mathbb{R}$, there are **no** conjugate points along any geodesic; but no geodesic that warps more than halfway around the cylinder is minimizing.

Proof. Given $\xi \in \partial B_1(0_p)$, $c(\xi) < \infty$, $c(\xi)\xi \in \mathcal{E}$, consider a strictly decreasing sequence $\{\epsilon_j\}$ with $\epsilon_j > 0$ for all j and $\epsilon_j \searrow 0$, as $j \to \infty$.
By the Hopf-Rinow theorem, there exists a sequence of geodesics emanating from p, with initial unit velocity vectors $\eta_j \in \partial B_1(0_p)$, and respective lengths $c(\xi) + \epsilon'_j > 0$, $\epsilon'_j < \epsilon_j$ such that

$$\gamma_\xi(c(\xi) + \epsilon_j) = \gamma_{\eta_j}(c(\xi) + \epsilon'_0),$$

$$c(\xi) + \epsilon'_j = d(p, \gamma_{\eta_j}(c(\xi) + \epsilon'_j)) < c(\xi) + \epsilon_j;$$

that is, γ_{η_j} minimizes distance-strictly less than $c(\xi) + t\epsilon_j$-from p to $\gamma_\xi(c(\xi) + \epsilon_j)$ for all j.
– It is impossible that infinitely many η_j denote the same element of $\partial B_1(0_p)$.
— Then $\{\eta_j\}$ has a convergent subsequence $\{\zeta_k\} = \{\eta_k\}$ with $\zeta_k \to \zeta$ as $k \to \infty$.
(2) If $\zeta \neq \xi$, then $d_{j_k} \to c(\xi)$ and $\gamma_\zeta(c(\xi)) = \gamma_\xi(c(\xi))$.
Thus there are at least two distinct minimizing geodesics from p to $\gamma_\xi(c(\xi))$.
(1) If $\zeta = \xi$, **claim:** $\exp_p$ **has a critical point at** $c(\xi)\xi$, i.e. $d\exp_p$ **is singular at** $c(\xi)\xi$**,** which implies $\gamma(c(\xi))$ is conjugate to p along γ_ξ—the first possibility.
– Suppose that $d\exp_p$ is not singular at $c(\xi)\xi$, then there exists a nbhd U of $c(\xi)\xi$ where $\exp_p$ is a diffeomorphism.
– Take ϵ_j sufficiently small so that $()\eta_j \in U$ and $()\xi \in U$. Then

$$\exp_p((c(\xi) + \epsilon_j)\xi) = \gamma(c(\xi) + \epsilon_j) = \sigma_j(c(\xi) + \epsilon'_j) = \exp_p((c(\xi) + \epsilon'_j)\eta_j)$$

hence $(c(\xi) + \epsilon_j)\xi = (c(\xi) + \epsilon'_j)\eta_j$, and hence $\xi = \eta_j$, which contradicts the definition of $c(\xi)$. □

Theorem 4. *The domain* $\mathbf{D}_p$ *is the largest domain, starlike w.r.t. the origin of* M_p*, for which* $\exp_p$ *restricted to that domain is a diffeomorphism. Furthermore,*

$$D_p = M \setminus C(p).$$

Definition. *Given any* $p \in M$*, define the* **injectivity radius** *of* p *by*

$$\operatorname{inj} p = \inf\{c(\xi) : \xi \in \partial B_1(0_p)\};$$

the **injectivity radius** *of* M, $\operatorname{inj} M$*, is defined by*

$$\operatorname{inj} M = \inf\{\mathit{inj}\, p : p \in M\}.$$

- Theorem 4 shows that $D_p = M \setminus \mathrm{Cut}_p(t)$ is homeomorphic to to an open ball of Euclidean space.
  In a certain cense, this indicates that the topology of M is contained in its cut locus.

Proposition 5. *If* M *is complete, then for all unit tangent vectors* $\xi \in TM$*, for which we have* $c(\xi) < \infty$*, we also have*

$$c(-\gamma'(c(\xi))) = c(\xi).$$

Corollary 6. *If* M *is complete, then for* p*,* $q \in M$*, we have* $q \in C(p)$ *iff* $p \in C(q)$*.*

Notation. *In what follows, we let* $\mathbf{S}M$ *denote the* **unit tangent bundle** *of* M*, that is,*

$$\mathbf{S}M = \{\xi \in TM : |\xi| = 1\},$$

with the natural projection $\pi\Big|_{\mathbf{S}M}$*, where* π *denotes the projection of* TM *on* M*.*

Theorem 7. *(1) The function* $c : \mathbf{S}M \to (0, \infty]$*, where* c *is the distance along* γ_ξ *from* $\pi(\xi)$ *to the cup point of* $\pi(\xi)$ *along* γ_ξ*, is upper semicontinuous on* $\mathbf{S}M$*.*
(2) If M *is Riemannian complete, then the function* c *is continuous on* $\mathbf{S}M$*.*

Proof. (1) Suppose we are given $\xi \in \mathbf{S}M$, with a sequence (ξ_k) in $\mathbf{S}M$ for which $\xi_k \to \xi$ as $k \to \infty$. Set

$$p = \pi(\xi), \quad p_k = \pi(\xi_k). \;\; d_k = c(\xi_k).$$

(i) If the sequence (d_k) has an unbounded subsequence $(\delta_j) = (d_{k_j})$ for which $\delta_j \nearrow \infty$ as $j \to \infty$, then for $T > 0$, one has, for sufficiently large j, $\delta_j > T$. Then

$$\lim_{j\to\infty} \gamma_{\xi_j}(T) = \gamma_\xi(T),$$

and

$$d(p, \gamma_\xi(T)) = \lim_{j\to\infty} d(p_{k_j}, \gamma_{\xi_j}(T)) = T.$$

Hence $c(\xi) = +\infty$.

(ii) Similarly, if (d_k) has a convergent subsequence

$$(\delta_j) = (d_{k_j}) \to \delta \quad \text{as } j \to \infty,$$

then again one has for all positive $\epsilon < \delta$

$$\begin{aligned} d(p, \gamma_\xi(\delta - \varepsilon)) &= \lim_{j\to\infty} d(p_{k_j}, \gamma_{\xi_j}(\delta_j - \varepsilon)) \\ &= \lim_{j\to\infty} \delta_j - \varepsilon \\ &= \delta - \varepsilon. \end{aligned}$$

Hence $c(\xi) \geq \delta$. In sum, we have

$$\limsup_{k\to\infty} c(\xi_k) \leq c(\xi).$$

(2) It remains to show that if M is complete, then

$$(*) \qquad \liminf_{k\to\infty} c(\xi_k) \geq c(\xi).$$

It suffices to assume that the sequence $c(\xi_k) \to \delta < +\infty$ as $k \to \infty$.
Thus, we **claim: γ_ξ cannot minimize past $\gamma_\xi(\delta)$.**
By passing to a subsequence if necessary, we may assume that either

(i) $\gamma_{\xi_k}(c(\xi_k))$ is conjugate to p_k along γ_{ξ_k} for all k, or
(ii) to each k one has $\eta_k \in \mathbf{S}M$, $\eta_k \neq \xi_k$ for all k, for which

$$\pi(\eta_k) = \pi(\xi_k) = p_k \quad \text{and} \quad \gamma_{\eta_k}(c(\eta_k)) = \gamma_{\xi_k}(c(\xi_k)), \quad \forall k.$$

— In **Case (i)**, $\gamma_\xi(\delta)$ is certainly conjugate to p along γ_ξ; so $c(\xi) \leq \delta$.
— In **Cae (ii)**, by passing to a subsequence if necessary, we may assume the existence of $\eta \in \mathbf{S}M$ for which $\eta_k \to \eta$ as $k \to \infty$.
But then,

$$\pi(\eta) = \pi(\xi) = p \quad \text{and} \quad \gamma_\eta(\delta) = \gamma_\xi(\delta).$$

(a) If $\eta \neq \xi$, then certainly $c(\xi) \leq \delta$.
(b) If $\eta = \xi$, then the map $\pi \times \exp$ is not a doffeomorphism on a neighborhood of $(\delta\xi, \delta\xi)$ in $TM \times TM$.
This implies $\gamma_\xi(\delta)$ is conjugate to p along γ_ξ.
Again, we have $c(\xi) \leq \delta$. This concludes the proof of (*). □

Theorem 7. *The function* inj: $M \to (0, +\infty]$ *is continuous.*

Theorem 8 (Klingenberg's lemma (1959)). *Let M be a complete Riemannian manifold, $p \in M$ and $q \in C(p)$ such that*

$$d(p,q) = d(p, C(p)),$$

that is, q is the point in $C(p)$ closest to p.
- *If q is not conjugate to p along a minimizing geodesic connecting p to q, then q is the midpoint of a geodesic loop, starting and ending at p.*
- *In particular, if M is compact and the sectional curvature of M satisfy*

  $$\mathcal{K} \leq \delta,$$

  then

  $$\operatorname{inj} M \geq \min\{\pi/\sqrt{\delta}, \ell(M)/2\},$$

  where $\ell(M)$ is the length of the shortest simple closed geodesic in M.

Proof. Given p and q as described previously,
if q is not conjugate to p along any minimizing geodsic connecting p to q,
then there exist two distinct unit speed minimizing geodesic segments γ_1 and γ_2 from p to q.
Neither contain any points conjugate to p.
- Let L denote the common length of γ_1 and γ_2,

  $$\gamma_1(0) = \gamma_2(0) = p.$$

  Then, one has the two surfaces given by

  $$\{\gamma_\xi(L)\} \quad \text{and} \quad \{\gamma_\eta(L)\}$$

  where ξ vary over a nbhd of $\gamma_1'(0)$ in $\partial B_1(0_p)$, and η varies over a nbhd of $\gamma_2'(0)$ in $\partial B_1(0_p)$.
- If $\gamma_1'(L) \neq -\gamma_2'(L)$, then the two hypersurfaces intersect transversally at q. This implies that, for varying ξ and η,

  $$\{\gamma_\xi(L-\epsilon)\} \cap \{\gamma_\eta(L-\epsilon)\}$$

  for sufficently small $\epsilon > 0$, which contradicts the assumption that q is the point in $C(p)$ closest to p. $\square$