Differential Forms

The Algebra of Alternating Tensors

Definition. *For any finite-dimensional real vector space V, let $\Lambda^k(V)$ denote the subspace of $T^k(V)$ consisting of alternating tensors (k-covectors).*

- There are no nonzero alternating k-tensors on V if $k >\dim V$.
- Every 0-tensor (which is a real number) is alternating, because there is no argument to interchange.
- Similarly, every 1-tensor is alternating.
- An alternating 2-tensor is just a skew-symmetric bilinear form on V.

— Note that any 2-tensor T can be expressed as the sum of an alternating tensor and a symmetric one, because

$$\begin{aligned} T(X,Y) =& \frac{1}{2}(T(X,Y) - T(Y,X)) + \frac{1}{2}(T(X,Y) + T(Y,X)) \\ =& A(X,Y) + S(X,Y), \end{aligned}$$

where

$$A(X,Y) = \frac{1}{2}(T(X,Y) - T(Y,X))$$

is alternating, and

$$S(X,Y) = \frac{1}{2}(T(X,Y) + T(Y,X))$$

is symmetric.

- Let S_k =the group of permutations of the set $\{1, \cdots, k\}$.
  Given a k tensor T ans a permutation $\sigma \in S_k$, define a new tensor ${}^{\sigma}T$ by
  $${}^{\sigma}T = T(X_{\sigma(1)}, \cdots, T_{\sigma(k)}).$$
- The tensor S defined above is $\mathrm{Sym} T \equiv \frac{1}{k!}\sum_{\sigma\in S^k} {}^{\sigma}T$, the symmetrization of T.

Definition. $\mathrm{Alt} : T^k(V) \to \Lambda^k(V)$, *called the* **alternating projection**, *is defined as follows:*

$$\mathrm{Alt}\, T = \frac{1}{k!}\sum_{\sigma\in S_k} (sgn\,\sigma)({}^{\sigma}T).$$

More explicitly, this means

$$\mathrm{Alt}\, T(X_1, \cdots, X_k) = \frac{1}{k!}\sum_{\sigma\in S_k} (sgn\,\sigma) T(X_{\sigma(1)}, \cdots, X_{\sigma(k)}).$$

Examples. If T is any 1-tsnsor, then $\mathrm{Alt}\, T = T$.

— If T is a 2-tensor, then $\mathrm{Alt}\, T(X,Y) = \frac{1}{2}(T(X,Y) - T(Y,X))$.

— For a 3-tensor T,

$$\begin{aligned} \mathrm{Alt}\, T(X,Y,Z) =& \frac{1}{6}(T(X,Y,Z) + T(Y,Z,X) + T(Z,X,Y) \\ & - T(Y,X,Z) - T(X,Z,Y) - T(Z,Y,X)). \end{aligned}$$

Lemma 3 (Properties of the Alternating Properties).
 (a) For any tensor T, $\mathrm{Alt}\, T$ *is alternating.*
 (b) T is alternating iff $\mathrm{Alt}\, T = T$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Elementary Alternating Tensors

Definition. *Let $k \in \mathbb{N}$. An ordered k-tuple $I = (i_1, \cdots, i_k)$ of positive integers is called a* **multi-index** *of length k.*

- *If I is such a multi-index and $\sigma \in S_k$ is a permutation, we write I_σ for the multi-index*

$$I_\sigma = (i_{\sigma(1)}, \cdots, i_{\sigma(k)}).$$

- Note that $I_{\sigma\tau} = (I_\sigma)_\tau$, for $\sigma, \tau \in S_k$.
- If I and J are multi-indices of length k, we define

$$\delta_I^J = \begin{cases} \operatorname{sgn}\sigma & \text{if neither } I \text{ nor } J \text{ has a repeated index} \\ & \qquad \text{and } J = I_\sigma \text{ for some } \sigma \in S_k, \\ 0 & \text{if } I \text{ or } J \text{ has a repeated index} \\ & \qquad \text{or } J \text{ is not a permutation of } I. \end{cases}$$

- Let V be an n-dimensional vector space, and suppose $(\varepsilon^1, \cdots, \varepsilon^n)$ is any basis for V^*.

— We will define a collection of alternating tensors on V that **generalize the determinant function on $\mathbb{R}^n$**.

— For each multi-index $I = (i_1, \cdots, i_k)$ of length k such that $1 \le i_1, \cdots, i_k \le n$, define a covariant k-tensor ε^I by

$$\varepsilon^I(X_1, \cdots, X_k) = \det \begin{pmatrix} \varepsilon^{i_1}(X_1), & \cdots, & \varepsilon^{i_1}(X_k) \\ \vdots & & \vdots \\ \varepsilon^{i_k}(X_1), & \cdots, & \varepsilon^{i_k}(X_k) \end{pmatrix} = \det \begin{pmatrix} X_1^{i_1} & \cdots & X_k^{i_k} \\ \vdots & & \vdots \\ X_1^{i_k} & \cdots & X_k^{i_k} \end{pmatrix} \tag{12.1}$$

In other words, if $\mathbb{X}$ denotes the matrix whose columns are the components of the vectors $X_1, \cdots, X_k$ w.r.t. the basis (E_i) dual to (ε^i), then $\varepsilon^I(X_1, \cdots, X_k)$ is the determinant of the $k \times k$ minor consisting of rows $i_1, \cdots, i_k$ of X.

- Because the determinant changes sign whenever two columes are interchanged, it is clear that ε^I is an alternating k-tensor.

Definition. ε^I *is called an* **elementary alternating tensor** *or* **elementary k-covector**.

Example. In terms of the standard dual basis $(e^1, e^2.e^3)$ for $(\mathbb{R}^3)^*$, we have

$$\begin{aligned} \varepsilon^{13}(X, Y) &= X^1 Y^3 - Y^1 X^3; \\ \varepsilon^{123}(X, Y, Z) &= \det(X, Y, Z). \end{aligned}$$

Lemma 4. *Let (E_i) be a basis for V, let (ε^i) be the dual basis for V^*, and let ε^I be as defined above.*

(a) If I has a repeated index, then $\varepsilon^I = 0$.
(b) If $J = I_\sigma$ for some $\sigma \in S_k$, then $\varepsilon^I = (\operatorname{sgn}\sigma)\varepsilon^J$.
(c) $\varepsilon^I(E_{j_1}, \cdots, E_{j_k}) = \delta^I_J$.

- The significance of the elementary k-covectors is that they provide a convenient basis for $\Lambda^k(V)$.

Definition. *A multi-inde $I = (i_1, \cdots, i_k)$ is said to be* **increasing** *if $i_1 < \cdots < i_k$.*

- It will be useful to use a primed summation sign to denote a sum over only increasing multi-indices, so that, for example,

$$\sum_I{}' T_I \varepsilon^I = \sum_{\{I: 1 \leq i_1 < \cdots < i_k \leq n\}} T_I \varepsilon^I.$$

Proposition 5. *Let V be an n-dimensional vector space. If (ε^i) is any basis for V^*, then for each positive integer $k \leq n$, the collection of k-covectors*

$$\mathcal{E} = \{\varepsilon^I : I \text{ is an increasing multi-index of length } k\}$$

is a basis for $\Lambda^k(V)$. Therefore

$$\dim \Lambda^k(V) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof. (I) If $k > n$, then $\Lambda^k(V)$ is the trivial vector space, since any k vectors are linearly dependent.

(II) If $k \leq n$, we need to show that $\mathcal{E}$ spans $\Lambda^k(V)$ and is independent.

— Let (E_i) be the basis for V dual to (ε^i).

(i) **To show that $\mathcal{E}$ spans** $\Lambda^k(V)$, let $T \in \Lambda^k(V)$ be arbitrary.
For each multi-index $I = (i_1, \cdots, i_k)$, define a real number T_I by

$$T_I = T(E_{i_1}, \cdots, E_{i_k}).$$

Since T is alternating

$$T_I = 0, \quad \text{if } I \text{ contains a repeated multi-index},$$

and

$$T_J = (\operatorname{sgn}\sigma) T_I, \quad \text{if } J = I_\sigma \text{ for } \sigma \in S_k.$$

For any multi-index J, Lemma 4 thus gives

$$\sum_I{}' T_I \varepsilon^I(E_{j_1}, \cdots E_{j_k}) = \sum_I{}' T_I \delta^I_J = T_J = T(E_{j_1}, \cdots E_{j_k}).$$

Therefore $\sum_I' T_I \varepsilon^I = T$. Hence $\mathcal{E}$ spans $\Lambda^k(V)$.

(ii) **To show that $\mathcal{E}$ is an independent set**, suppose

$$\sum_I{}' T_I \varepsilon^I = 0, \quad \text{for some coefficient } T_I.$$

Let J be any increasing multi-index. Applying both sides to $(E_{j_1}, \cdots E_{j_k})$ and using Lemma 4,

$$0 = \sum_I{}' T_I \varepsilon^I (E_{j_1}, \cdots E_{j_k}) = T_J.$$

Thus each coefficient $T_J = 0$. $\square$

- In particular, for an n-dimensional vector space V, this proposition implies that $\Lambda^n(V)$ is 1-dimensional and is spanned by $\varepsilon^{1\cdots n}$.

— By definition, this elementary n-covector acts on vectors $(X_1, \cdots, X_n)$ by taking the determinant of their component matrix $\mathbb{X} = (X^i_j)$.

— For example, on $\mathbb{R}^n$ with the standard basis, $\varepsilon^{1\cdots n}$ is precisely the determinant function.

Lemma 6. *Suppose V is an n-dimensional vector space and $\omega \in \Lambda^n(V)$. If $T : V \to V$ is any linear map and $X_1, \cdots, X_n$ are arbitrary vectors in V, then*

$$\omega(TX_1, \cdots, TX_n) = (\det T)\omega(X_1, \cdots, X_n). \tag{2}$$

Proof. Let (E_i) be any basis for V, and let (ε^i) be the dual basis.
Since both sides of (2) are multilinear functions of $X_1, \cdots, X_n$, it suffices to verify it in the special case $X_i = E_i$, $i = 1, \cdots, n$.

— By Proposition 5, we can write

$$\omega = c\varepsilon^{1,\cdots n}, \quad \text{for some } c \in \mathbb{R}.$$

Hence, we have

$$(\det T)\omega(E_1, \cdots, E_n) = (\det T)c\varepsilon^{1,\cdots n}(E_1, \cdots, E_n) = c \det T.$$

— On the other hand, letting (T^j_i) denote the matrix of T w.r.t. this basis, and letting

$$T_i = TE_i,$$

we have

$$\omega(TE_1, \cdots, TE_n) = c\varepsilon^{1,\cdots n}(T_1, \cdots, T_n) = c \det(\varepsilon^j(T_i)) = c \det(T^j_i). \quad \square$$

The Wedge Product

Definition. *If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$, we define the* **wedge product** *or* **exterior product** *of ω and η to be the alternating $(k+\ell)$-tensor*

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\ell!}\mathrm{Alt}\,(\omega \otimes \eta). \tag{12.3}$$

Lemma 7. *Let $(\varepsilon^1, \cdots, \varepsilon^n)$ be a basis for V^*. For any multi-indices $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_\ell)$,*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}, \tag{4}$$

where IJ is the multi-index $(i_1, \cdots, i_k, j_1, \cdots, j_\ell)$ obtained by contaternating I, J.

Proof. By multiplicity, it suffices to show that

$$\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \cdots, E_{p_{k+\ell}}) = \varepsilon^{IJ}(E_{p_1}, \cdots, E_{p_{k+\ell}}) \tag{5}$$

for any sequence $(E_{p_1}, \cdots, E_{p_{k+1}})$ of basis vectors. We consider several cases:
Case I: $P = (p_1, \cdots, p_{k+\ell})$ has a repeated index.
In this case, both sides of (5) are zero.

Case II: P contains an index that does not appear on either I or J.
In this case, both sides of (5) are zero by Lemma 4(c).
Case III: $P = IJ$ and P has no repeated indices.
We need to show that the left-hand side is also equal to 1. By definition,

$$\begin{aligned}
&\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \cdots, E_{p_{k+\ell}})\\
=&\frac{(k+\ell)!}{k!\ell!}\mathrm{Alt}(\varepsilon^I \otimes \varepsilon^J)(E_{p_1}, \cdots, E_{p_{k+\ell}})\\
=&\frac{1}{k!\ell!}\sum_{\sigma \in S_{k+\ell}} (\mathrm{sgn}\sigma)\varepsilon^I(E_{p_{\sigma(1)}}, \cdots, E_{p_{\sigma(k)}})\varepsilon^J(E_{p_{\sigma(k+1)}}, \cdots, E_{p_{\sigma(k+\ell)}}).
\end{aligned}$$

By Lemma 4 again, the only terms in the sum above that give nonzero values are those in which σ permutes the first k indices and the last ℓ indices of P seperately.

- In other words, σ must be of the form $\sigma = \tau\eta$, where $\tau \in S_k$ acts by permutes $\{1, \cdots, k\}$ and $\eta \in S_\ell$ acts by permuting $\{k+1, \cdots, k+\ell\}$.
- Since $\mathrm{sgn}(\tau\eta) = (\mathrm{sgn}\,\tau)(\mathrm{sgn}\,\eta)$, we have

$$\begin{aligned}
&\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \cdots, E_{p_{k+1}})\\
=&\frac{1}{k!\ell!}\sum_{\substack{\tau \in S_k,\\ \eta \in S_\ell}} (\mathrm{sgn}\,\tau)(\mathrm{sgn}\,\eta)\varepsilon^I(E_{p_{\tau(1)}}, \cdots, E_{p_{\tau(k)}})\varepsilon^J(E_{p_{\eta(k+1)}}, \cdots, E_{p_{\eta(k+\ell)}})\\
=&\left(\frac{1}{k!}\sum_{\tau \in S_k}(\mathrm{sgn}\,\tau)\varepsilon^I(E_{p_{\tau(1)}}, \cdots, E_{p_{\tau(k)}})\right)\left(\frac{1}{\ell!}\sum_{\eta \in S_\ell}(\mathrm{sgn}\,\eta)(\mathrm{Alt}\varepsilon^J)(E_{p_{k+1}}, \cdots, E_{p_{k+\ell}})\right)\\
=&(\mathrm{Alt}\,\varepsilon^I)(E_{p_1}, \cdots, E_{p_k})(\mathrm{Alt}\,\varepsilon^J)(E_{p_1}, \cdots, E_{p_{k+\ell}})\\
=&\varepsilon^I(E_{p_1}, \cdots, E_{p_k})\varepsilon^J(E_{p_1}, \cdots, E_{p_{k+\ell}}) = 1.
\end{aligned}$$

∎

Case IV: P is a permutation of IJ and P has no repeated indices.
In this case, applying a permutation of P brings us back to Case III.
Since the effect of the permutation is to multiply both sides of (5) by the same sign, the result holds in this case as well. □

Proposition 8 (Properties of the Wedge Product).

(a) Bilinearity*:* $(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta),$

$$\eta \wedge (a\omega + a'\omega) = a(\eta \wedge \omega) + a'(\eta \wedge \omega').$$

(b) Associativity*:* $\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi.$

(c) Anticommutativity: *For* $\omega \in \Lambda^K(v)$ *and* $\eta \in \Lambda^\ell(V)$, $\omega \wedge \eta = (-1)^{k\ell}\eta \wedge \omega.$

(d) If $(\varepsilon^1, \cdots, \varepsilon^n)$ *is any basis for* V^* *and* $I = (i_1, \cdots, i_k)$ *is any multiindex, then*

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} = \varepsilon^I. \tag{7}$$

(e) For any covectors $\omega^1, \cdots, \omega^k$ *and vectors* $X_1, \cdots, X_k$,

$$\omega^1 \wedge \cdots \omega^k(X_1, \cdots, X_k) = \det(\omega^j(X_i)). \tag{8}$$

Proof. (a) immediately follows from the definition, because the tensor product is bilinear and Alt is linear.

(b) To prove associativity, note that Lemma 7 gives

$$(\varepsilon^I \wedge \varepsilon^J) \wedge \varepsilon^K = \varepsilon^{IJ} \wedge \varepsilon^K = \varepsilon^{IJK} = \varepsilon^I \wedge \varepsilon^{JK} = \varepsilon^I \wedge (\varepsilon^J \wedge \varepsilon^J).$$

The general case follows from bilinearity.

(c) Using Lemma 7 again, we obtain

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ} = (\mathrm{sgn}\tau)\varepsilon^{JI} = (\mathrm{sgn}\tau)\varepsilon^J\varepsilon^I,$$

where τ is the permutation that sends IJ to JI. We have

$$\mathrm{sgn}\,\tau = (-1)^{k\ell},$$

since τ can be decomposed as a composition of $k\ell$ transpositions (each index of I must be moved past each of the indices of J).

Anticommutativity then follows from bilinearity.

(d) is an immediate consequence of Lemma 7 and induction.

(e) To prove (e), we note that the special case in which each ω^j is one of the basis covectors ε^{i_j} reduces to (7).

Since both sides of (8) are multilinear in $(\omega^1, \cdots, \omega^k)$, this suffices. □

- For any n-dimensional vector space V, define a vector space $\Lambda^*(V)$ by

$$\Lambda^*(V) = \bigoplus_{k=0}^{n} \Lambda^k V.$$

- It follows from Proposition 5 that $\dim\Lambda^*(V) = 2^n$.
- Proposition 8 shows that the wedge product turns $\Lambda^*(V)$ into an associative algebra, called the **exterior algebra** of V.
- This algebra is **not** commutative, but it has a closely related property.

Definition. *An algebra A is said to be* **graded** *if it has a direct sum decomposition* $A = \bigotimes_k A^k$ *such that the product satisfies*

$$(A^k)(A^\ell) \subset A^{k+\ell}.$$

- *A graded algebra is* **anticommutative** *if the product satisfies* $ab = (-1)^{k\ell}ba$ *for* $a \in A^k$, $b \in A^\ell$.

- Proposition 8(c) shows that $\Lambda^*(V)$ is an anticommutative graded algebra.

Differential Forms on Manifolds

Definition. *The subset of T^kM consisting of alternating tensors is denoted by $\Lambda^k M$:*

$$\Lambda^k M = \coprod_{p\in M} \Lambda^k(T_pM).$$

Definition. *A section of $\Lambda^k M$ is called a* **differential k-form**, *or a* **k-form**; *this is a (continuous) tensor field whose value at each point is an alternating tensor.*
- *The integrer k is called the* **degree** *of the form.*

Definition. *Denote the vector space of smooth sections of $\Lambda^k M$ by $\mathcal{A}^k(M)$.*

- In any smooth chart, a k-form ω can be written locally as

$$\omega = \sum_I{}' \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_I{}' \omega_I dx^I,$$

  where the coefficients ω_I are continuous functions defined on the coordinate domain, and we use dx^I as the abbreviation for $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.
- In terms od differential forms, the result of Lemma 4(c) translates to

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}\Big(\frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_k}}\Big) = \delta^I_J.$$

  Thus the component functions ω^I of ω are determined by

$$\omega^I = \omega\Big(\frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_k}}\Big).$$

- A 0-form is just continuous real-valued function.
- A 1-form is a covector field.

Definition. *The wedge product of two differential forms is defined pointwise:*

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p.$$

- Thus the wedge product of a k-form with an ℓ-form is a $(k+\ell)$-form.
- If f is a 0-form and η is a k-form, we interpret the wedge product$f \wedge \eta$ to mean the ordinary product $f\eta$.
- If we define

$$\mathcal{A}^*(M) = \bigoplus_{k=0}^{n} \mathcal{A}^k(M), \tag{11}$$

  the wedge product turns $\mathcal{A}^*(M)$ into an associative, anticommutative graded algebra.

Definition. *If $F : M \to N$ is a smooth map and ω is a smooth differential form on N, the pullback $F^*\omega$ is a smooth differential form on M, defined as for any smooth tensor field:*

$$(F^*\omega)_p(X_1, \cdots, X_k) = \omega_{F(p)}(F_*X_1, \cdots, F_*X_k).$$

- *If $\imath : N \hookrightarrow M$ is the inclusion map of an immersed submfd, then we use the notation $\omega|_N$ for $\imath^*\omega$.*

Lemma 10. *Suppose $F : M \to M$ is smooth.*
(a) $F^* : \mathcal{A}^k(N) \to \mathcal{A}^k(M)$ is linear.
(b) $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.
(c) In any smooth chart,

$$F^*(\sum_I{}' \omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}) = \sum_I{}' (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F).$$

Proposition 12. *Let $F : M \to N$ be a smooth map between n-manifolds. If (x^i) and (y^j) are smooth coordinates on open sets $U \subset M$ and $V \subset N$, respectively, and u is a smooth real-valued function on V, then the following holds on $U \cap F^{-1}(V)$:*

$$F^*(u dy^1 \wedge \cdots \wedge dy^n) = (u \circ F)(\det DF) dx^1 \wedge \cdots \wedge dx^n, \tag{12}$$

where DF represents the matrix of partial derivatives of F in cooedinates.

Proof. It suffices to show that both sides of (12) give the same result when evaluated on $(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x^n})$. From Lemma 10,

$$F^*(u dy^1 \wedge \cdots \wedge dy^n) = (u \circ F) dF^1 \wedge \cdots \wedge dF^n.$$

Proposition 8(e) shows that

$$dF^1 \wedge \cdots \wedge dF^n \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x^n} \right) = \det \left(dF^j (\frac{\partial}{\partial x^i}) \right) = \det \left(\frac{\partial F^j}{\partial x^i} \right).$$

Therefore

$$F^*(u dy^1 \wedge \cdots \wedge dy^n) \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x^n} \right) = (u \circ F) \det DF. \quad \square$$

Corollary 13. *If $(U, (x^i))$ and $(\widetilde{U}, (\tilde{x}^j))$ are overlapping smooth coordinate charts on M, then the following identity holds in $U \cap \widetilde{U}$:*

$$d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n = \det \left(\frac{\partial \tilde{x}^j}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^n. \tag{13}$$

Proof. Apply Proposition 12 with F equal to the identity map of $U \cap \widetilde{U}$, but using coordinates (x^i) in the domain and $(\widetilde{x}^j)$ in the range. $\square$