**The Hodge Theorem**

In this section we assume that $M$ is an oriented compact Riemannian manifold without boundary.

Since every hamnonic form is closed (by Proposition 1 (iii)), we obtain a linear map

$$\mathbb{H}^k(M) \to H^k_{dR}(M)$$

by taking the de Rham cohomology.

**Lemma.** *The map*

$$\mathbb{H}^k(M) \to H^k_{dR}(M)$$

*is an injection.*

*Proof.* It suffices to **claim: if a harmonic form $\omega$ is exact, then** $\omega = 0$.
Indeed, if $\omega = d\eta$, then by Proposition 10, we obtain

$$(\omega, \omega) = (d\eta, \omega) = (\eta, \delta\omega) = (\eta, 0) = 0.$$

Hence $\omega = 0$. □

- From de Rham's theorem that $H^k_{dR}(M)$ is isomorphic to $H^k(M;\mathbb{R})$; hence $H^k_{dR}(M)$ is finite-dimensional.

— Combining this and Lemma 3, we see that $\mathbb{H}^k(M)$ is also finite-dimensional.
— In fact, we have indeed the following result.

**Hodge theorem.** *An arbitrary de Rham cohomology class of an oriented compact Riemannian manifold can be represented by a unique harmonic form.*
*In other words, the natural map $\mathbb{H}^k(M) \to H^k_{DR}(M)$ is an isomorphism.*

The essence of this theorem lies in the assertion on the **existence** of a harmonic form, and existence theorems are in general difficult.

**Proof of the Hodge Theorem bases on the Hodge Decomposition.**

It suffices to **claim: the natural map $\mathbb{H}^k(M) \to H^k_{DR}(M)$ is surjective.**
Let $\omega \in \mathcal{A}^k(M)$ be any closed form and let

$$\omega = \omega_H + d\eta + \delta\theta$$

be the Hodge decomposition of $\omega$. We have

$$\omega_H = H\omega.$$

By assumption, we have

$$0 = d\omega = d\delta\theta.$$

$$\therefore 0 = (d\delta\theta, \theta) = (\delta\theta, \delta\theta).$$

$$\therefore \delta\theta = 0.$$

$$\therefore \omega = \omega_H + d\eta.$$

Thus $\omega$ is cohomologous to the harmonic form $\omega_H$, as we wanted to show. □

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

**Applications of The Hodge Theorem**

**(a) The Poincaré duality theorem**

Let $M$ be a connected, compact oriented $n$-dimensional $C^\infty$ manifold.

— For each $k$ $(0 \leq k \leq n)$, we define a bilinear map

$$H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{R}$$

by setting

$$([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta,$$

where *omega* and $\eta$ are closed $k$- and $(n-k)$-forms, and $[\omega]$ and $[\eta]$ the de Rham cohomology classes represented by $\omega$ and $\eta$, respectively.

– This map is obviously bilinear.

— That the image **is independent of the choice of closed forms** representing the de Rham cohomology classes follows from Stokes' theorem, namely,

$$\begin{aligned}\int_M (\omega + d\alpha) \wedge (\eta + d\beta) &= \int_M \omega \wedge \eta + \int_M d(\alpha \wedge \eta + (-1)^k \omega \wedge \beta + \alpha \wedge d\beta) \\ &= \int_M \omega \wedge \eta.\end{aligned}$$

**Poincaré duality theorem.** *For a connected, compact oriented $n$-dimensional $C^\infty$ manifold, thr bilinear map*

$$H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{R}$$

*defined above is* **nondegenerate** *and hence induces an isomorphism*

$$H^{n-k}_{dR}(M) \cong H^k_{dR}(M)^*.$$

*Proof.* **Non-degeneracy** of the map means that for any cohomology class $[\omega] \in H^k_{DR}(M)$, there exists a certain $[\eta] \in H^{n-k}_{DR}(M)$ such that $\int_M \omega \wedge \eta \neq 0$.

— In order to prove this, choose a Riemmanian metric.

– By the Hodge theorem we may assume that $\omega$ is a harmonic form relative to the metric that is not zero identically.

– If $\eta = *\omega$, then $\eta$ is also a harmonic form, which is closed.

– Since

$$\int_M \omega \wedge \eta = \|\omega\|^2 \neq 0,$$

we conclude the proof. □

**(b) Manifolds and Euler number.**

- Suppose a figure $K$ is triangulated with $\alpha_i$ as the number of $i$-dimensional simplices. Then the alternate sum

$$\sum_i (-1)^i \alpha_i$$

is an invariant regardless of the way $K$ is triangulated.

– This invariant is equal to the alternate sum of Betti numbers, namely

$$\chi(K) = \sum_i (-1)^i \beta_i, \quad \beta_i = \dim H_i(K; \mathbb{R}).$$

Here $\chi(K)$ is called the **Euler number** or **Euler characteristic** or **Euler-Poincaré characteisitic**.

— For an $n$-dimensional manifold $M$, we have

$$\chi(M) = \sum_i (-1)^i \dim H^i_{DR}(M).$$

- The next theorem is a simple application of the Poincaré duality theorem

**Theorem.** *The Euler characteristic of an odd-dimemsional closed manifold is* 0.

*Proof.* Although the theorem holds for any topological manifold, we shall prove it for $C^\infty$ manifold.

— It is clearly suffices to prove it for a **connected** manifold.

— Thus let $M$ be a $(2n+1)$-dimensional connected closed manifold.

— We may assume that $M$ is **oriented**;
indeed, if $M$ is non-orientable, let $\widetilde{M}$ be the set of all pairs $(p, \sigma)$,
where $p$ is a point on $M$ and $\sigma$ is an orientation in $T_pM$,
then $\widetilde{M}$ is a connected and orientable $C^\infty$ manifold
and the natural projection $\pi : \widetilde{M} \to M$ is a double covering map,
from which we find

$$\chi(\widetilde{M}) = 2\chi(M),$$

using the triangulation of $\widetilde{M}$ induced from that of $M$.

— In case $M$ is **oriented**, by the Poincaré duality theorem, we have

$$H^{2n+1-k}_{DR}(M) \cong (H^k_{DR}(M))^*, \quad \forall k.$$

$$\therefore \dim H^k_{DR}(M) = \dim (H^k_{DR}(M))^* = \dim H^{2n+1-k}_{DR}(M),$$

$$\therefore \chi(M) = \sum_i (-1)^i \dim H^i_{DR}(M) = 0. \qquad \square$$