Representation Theory: Elementary Concepts

Definition. *A (finite-dimensional)* **representation** *of a Lie group G is a homomorphism*

$$\phi : G \to \mathrm{Aut}(V),$$

where V is a (finite-dimensional) vector space.
The dimension of the representation is the dimension of the vector space V.
Denote the representation of G in V by (G, V) or simply by V.

- If (G, V) is a representation of G and $g \in G$, $v \in V$, then this defines an **action** $\Phi : G \times V \to G$ of G on V as follows:

$$\Phi(g, v) = \phi(g)(v) \overset{\text{denote}}{=} g \cdot v.$$

Then we obtain that $e \cdot v = v$ and $g_1 \cdot (g_2 \cdot v) = (g_1 \cdot g_2) \cdot v$.
For this reason a representation (G, V) is also referred to as a G-space.

- If the space V is real (respectively complex, or quaterion) vector space and, if for all $g \in G$, the maps

$$\Phi(g) : V \to V, \quad v \mapsto \Phi(g \cdot v)$$

  are linear, then the corresponding representation is called real (respectively complex, or quaterion).

Definition. *Let (G, V) be a representation. A subspace U of V is called* **invariant** *or G-invariant if*

$$g \cdot U \subset U, \quad \forall g \in G.$$

Lemma. *Let G be a compact Lie group, and $C(G)$ the set of all continuous real-valued functions on G. Then there exists a unique function $I : C(G) \to \mathbb{R}$ such that*
(a) $I(1) = 1$,
(b) I is positive (i.e. $I(f) \geq 0$ for $f \geq 0$) and linear,
(c) I is invariant; i.e. $I(f) = I(f \circ L_g) = I(R_g \circ f)$ for all $g \in G$.
The number $I(f)$ is denoted $\int_G f(g)dg$ and is called a **Harr integral** *on G. (It is usually realized by some of integration on G.)*

Theorem 0. *Let $\phi : G \to \mathrm{Aut}(V)$ be a representaion of a* **compact** *group G. Then there exists a G-invariant inner product $(,)$ on V, i.e.*

$$(g \cdot u, g \cdot v) = (u, v) \quad \forall u, v \in G, \quad g \in G.$$

Proof. Take an inner product $\langle , \rangle$ on V. Then define

$$(u, v) = \int_G \langle \phi(g)u, \phi(g)v \rangle \, dg, \quad \forall u, v \in V. \qquad \square$$

The Adjoint Representation

- The adjoint representation of a Lie group is a measure of the non-commutativity of the group.

Definition. *An* **automorphism** *of a Lie group G is a map $\phi : G \to G$ that is a diffeomorphism and a group isomorphism.*

Definition. *The map $I_g : G \to G$ sending each h to ghg^{-1} (*which is a homomorphism, and since $I_g = R_{g^{-1}} \circ L_g$ is a diffeomorphism*), is a called an* **inner automorphism** *of G.*

⊙ If G is abelian, then I_g is the identity map $h \mapsto h$, $\forall g \in G$.

- Notice that each I_g maps the identity e into itself, so that every curve through e is mapped into a (possibly different) curve through e.

– Thus I_g induces a map $(I_g)_*$ mapping any vector in T_e to another one in T_e.
– This map is called $\mathrm{Ad}(g)$, **the adjoint transformation of T_e induced by g.**

$$\mathrm{Ad}(g)(X_e) = (I_g)_*(X_e).$$

Definition. *The* **adjoint representation** *of G is the homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathrm{Lie}(G))$ given by*

$$\mathrm{Ad}(g) = (dI_g)_e; \quad \text{i.e. } \mathrm{Ad}(g)(X_e) = (I_g)_*(X_e).$$

This is a homomorphism since $I_{xy} = I_x \circ I_y$ implies that

$$\mathrm{Ad}(xy) = \mathrm{Ad}(x) \circ \mathrm{Ad}(y), \quad \text{by taking differentials.}$$

It is also smooth.
– Thus, $\sigma(t) = I_g(\exp(tX))$ is a one-parameter subgroup, with

$$\sigma'(0) = \frac{d}{dt}\bigg|_{t=0} I_g(\exp tX) = (I_g)_*(\frac{d}{dt}\bigg|_{t=0} \exp tX) = (I_g)_* X.$$

Hence

$$I_g(\exp tX) = \exp(t(I_g)_*X) = \exp(t\mathrm{Ad}(g)(X)).$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}(g)} & \mathfrak{g} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[I_g]{} & G. \end{array}$$

Proposition. $\mathrm{Ad}\,(g)(X) = (R_{g^{-1}})_* X$, $\forall g \in G$ *and* $X \in \mathrm{Lie}(G)$.

Proof. By definition

$$\mathrm{Ad}(g)X = dI_g(X) = (R_{g^{-1}})_*(L_g)_*(X) = (R_{g^{-1}})_*X, \ \ \forall g \in G \ \text{ and } \ X \in \mathrm{Lie}(G). \quad \square$$

- Now if g itself is a member of a one-parameter subgroup $g(s) = \exp(sY)$, there should be a natural expression for $\mathrm{Ad}(g)(X)$ in terms of Y. Indeed, we have

Theorem 1.

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(\exp tY)X = (\mathcal{L}_Y X)_e = [Y,X]\bigg|_e .$$

$$(\mathrm{Ad}(\exp tY))_* X = [Y,X].$$

Definition. *The* **adjoint representation** *of* Lie(G) *is the homomorphism* ad: Lie$(G) \to$ End(Lie(G)) *given by*

$$\mathrm{ad}(X) = (d\mathrm{Ad})_e(X) = \mathrm{Ad}_*(X_e).$$

Theorem 1*. $\mathrm{ad}(X)Y = [X,Y]$ *for all* X, $Y \in \mathrm{Lie}(G)$.

Proof. Let $x_t = \exp(tX)$ be the flow of $X \in \mathrm{Lie}(G)$. Since X is left-invariant

$$L_y \circ x_t = x_t \circ L_y \quad \forall y \in G,$$

which gives that

$$x_y(t) = x_t(L_y(e)) = L_y(x_t(e)) = R_{x_t(e)}(y),$$

and therefore

$$dx_t = dR_{x_t(e)}.$$

$$\begin{aligned}[X,Y] &= \lim_{t\to 0}\frac{1}{t}(Y - dx_t(Y)) = -\lim_{t\to 0}\frac{1}{t}(dR_{x_t(e)}(Y) - Y)\\ &= -\lim_{t\to 0}\frac{1}{t}(\mathrm{Ad}(x_t^{-1}(e))(Y) - Y), \quad \text{by (1)},\\ &= \lim_{t\to 0}\frac{1}{t}(\mathrm{Ad}(x_t(e))(Y) - Y) = \mathrm{ad}(X)Y. \qquad \square\end{aligned}$$

- From the Jacobi identity it follows that ad_X is a derivation

$$\mathrm{ad}(X)[Y,Z] = [\mathrm{ad}(X)(Y), Z] + [Y, \mathrm{ad}(X)(Z)].$$

From Theorem 1 it follows that

$$\mathrm{Ad}(\exp X) = \exp(\mathrm{ad}(X)).$$

Definition. $Z(G) = \{g \in G : gh = hg \quad \forall h \in G\}$ *denotes the* **center** *of* G *and* $Z(\mathrm{Lie}(G)) = \{X \in \mathrm{Lie}(G) : [X,Y] = 0 \ \ \forall Y \in \mathrm{Lie}(G)\}$ *denotes the center of* Lie(G).

Proposition 2. *Let* G *be a connected Lie group. Then*

$$\ker \mathrm{Ad} = Z(G) \quad \textit{and} \quad \ker \mathrm{ad} = Z(\mathrm{Lie}(G))$$

Furthermore, the Lie algebra of $Z(G)$ *is* $Z(\mathrm{Lie}(G))$.

- This theorem shows that the bracket operation in Lie(G) measures the failure of G to be commutative.

Definition. *A Lie algebra* $\mathfrak{g}$ *is called* **abelian** *if* $[X, Y] = 0$ *for all* X, $Y \in \mathfrak{g}$.

Corollary 3. *If* G *is abelian, then* $\mathrm{Lie}(G)$ *is abelian.*

Proof. If G is abelian, then I_g =Id, hence Ad_g =Id for all $g \in G$. Thus by the proof of the above theorem $[X, Y] = 0$ for all X, $Y \in \mathrm{Lie}(G)$. $\square$

- For the case of a **matrix group**(that is a subgroup of a general linear group), the adsjoint representation has a simple expression.

Proposition 4. *If* G *is a* **matrix group**, *then*

$$\mathrm{Ad}(g)X = gXg^{-1} \quad \forall g \in G, X \in \mathrm{Lie}(G),$$

(the multiplication being the multiplication of matrices).

Proof. Let $t \mapsto \exp(tX)$ be the one-parameter subgroup of X whose derivative at $t = 0$ is X.
Since G is a metrix group, the exponential map is given by the ordinary exponentiation of matrices, and thus we have

$$\begin{aligned} \mathrm{Ad}(g)X =& (I_g)_*(X_e) = \frac{d}{dt} I_g(\exp tX)\Big|_{t=0} = \frac{d}{dt} g(\exp tX)g^{-1}\Big|_{t=0} \\ =& g\frac{d}{dt}e^{tX}\Big|_{t=0} g^{-1} = gXg^{-1}. \qquad \square \end{aligned}$$

the Killing Forms

- We have seen that for any representation (G, V) of a compact Lie group G, there exists a G-invariant inner product on V.

– In paticular, this happens for the adjoint representation of $(G, \mathrm{Lie}(G))$.
– We will now introduce an explicit inner product on $\mathrm{Lie}(G)$.

Definition. *Given a Lie algebra* $\mathfrak{g}$, *define* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *by*

$$\mathrm{ad}X(Y) = [X, Y].$$

Definition. *The* **Killing form** *of a Lie algebra* $\mathfrak{g}$ *is the function* $B : \mathfrak{g} \times \mathfrak{g}(G) \to \mathbb{R}$ *given by*

$$B(X, Y) = \mathrm{tr}(\mathrm{adX} \circ \mathrm{ad}Y).$$

Proposition 5. *The Killing forms has the following properties:*

(1) *It is a symmetric bilinear form on* $\mathrm{Lie}(G)$.
(2) *If* $\mathfrak{g} = \mathrm{Lie}(G)$, *then* B **is** Ad**-invariant**, *i.e.*

$$B(X, Y) = B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y), \quad \forall g \in G, \quad X, Y \in \mathfrak{g}.$$

(3) *Each* $\mathrm{ad}(Z)$ *is* **skew-symmetric** *with respect to* B, *that is,*

$$B(\mathrm{ad}(Z)X, Y) = -B(X, \mathrm{ad}(Z)Y)$$

or

$$B([X, Z], Y) = B(X, [Z, Y]).$$

Proof. (1) Bilinearity follows from the linearity of $X \mapsto \mathrm{ad}(X)$ and the linearity of the trace. Symmetry follws from the fact that $\mathrm{tr}(AB)$=$\mathrm{tr}(BA)$.

(2) If $\sigma : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of $\mathfrak{g}$ (i.e. a linear isomorphism with $\sigma[X, Y] = [\sigma X, \sigma Y]$), then $\mathrm{ad}(\sigma X) \circ \sigma = \sigma \circ \mathrm{ad}(x)$, i.e.,

$$\mathrm{ad}(\sigma X) = \sigma \circ \mathrm{ad}(\sigma X) \circ \sigma^{-1}.$$

Take $\sigma = \mathrm{Ad}(g)$ and compute

$$\begin{aligned} B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) =& \mathrm{tr}(\mathrm{ad}(\mathrm{Ad}(g)X) \circ \mathrm{ad}(\mathrm{Ad}(g)Y)) \\ =& \mathrm{tr}(\mathrm{ad}(g) \circ \mathrm{ad}(X) \circ \mathrm{Ad}(g^{-1}) \circ \mathrm{Ad}(g) \circ \mathrm{ad}(Y) \circ \mathrm{Ad}(g)^{-1}) \\ =& \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = B(X, Y) \end{aligned}$$

(3) We use the Jacobi identity twice and obtain

$$\begin{aligned} [Z, [X, [Y, W]]] =& [[Z, X], [Y, W]] + [X, [Z, [Y, W]]] \\ =& [[Z, X], [Y, W]] + [X, [[Z, Y], W]] + [X, [Y, [Z, W]]]. \end{aligned}$$

Hence

$$\begin{aligned} &\mathrm{ad}(Z) \circ \mathrm{ad}(X) \circ \mathrm{ad}(Y) \\ &\quad = \mathrm{ad}(\mathrm{ad}(Z)X) \circ \mathrm{ad}(Y) + \mathrm{ad}(X) \circ \mathrm{ad}(\mathrm{ad}(Z)Y) + \mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \mathrm{ad}(Z), \end{aligned}$$

i.e.

$$[\mathrm{ad}(Z), \mathrm{ad}(X) \circ \mathrm{ad}(Y)] = \mathrm{ad}(\mathrm{ad}(Z)X) \circ \mathrm{ad}(Y) + \mathrm{ad}(X) \circ \mathrm{ad}(\mathrm{ad}(Z)Y).$$

Since $\mathrm{tr}([A, B]) = 0$, for all A, $B \in \mathfrak{g}$, we finally obtain that

$$B(\mathrm{ad}(Z)X, Y) + B(X, \mathrm{ad}(Z)Y) = 0. \qquad \square$$

Definition. *A Lie group G is called* **semisimple** *if the Killing form is nondegenerate.*

Proposition 8. *If G is semisimple, then the center $Z(\mathrm{Lie}(G)) = 0$.*

Proof. Let $X \in Z(\mathfrak{g})$. Then $[X, Y] = 0$ for all $Y \in \mathrm{Lie}(G)$, thus $\mathrm{ad}(X)$ is the zero operator, which gives that $B(X, X) = \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}X) = 0$. Since G is semisimple, $X = 0$. $\square$

Corollary 9. *The center of a semisimple Lie group is disctrete.*

Theorem 8. *If G is a compact semisimple Lie group, then its Killing form is negative definite.*

Proof. Since G is compact, by Theorem 0, there is an Ad-invariant inner product on $\mathrm{Lie}(G)$, so $\mathrm{Ad}(g)$ is an orthogonal transfornmation on $\mathrm{Lie}(G)$. Thus, $\mathrm{ad}\,X$ is skew-symmetric. Hence, if $\mathrm{ad}\,(X) = (a_{ij})$ relative to an orthonormal basis,

$$B(X, X) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(X)) = \sum_i \sum_j a_{ij} a_{ji} = -\sum_{ij} a_{ij}^2 \leq 0.$$

Since G is semisimple, B is nondegenerate, so the above sum is strictly less than zero. $\square$

A Note on Complexification.

- If V is a vector space over $\mathbb{R}$, then we can define the vector space

  $$V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} \quad \text{(or simply } V \otimes \mathbb{C}\text{)},$$

  whose dimension over $\mathbb{C}$ equals the dimension of V over $\mathbb{R}$.
  We can formally think of $V \otimes_{\mathbb{R}} \mathbb{C}$ as the set

  $$\{X + iY : X, Y \in V,\ i = \sqrt{-1}\}.$$

- If $\mathfrak{g}$ is a Lie algebra of $\mathbb{R}$, then the **complexification** of $\mathfrak{g}$ is the Lie algebra $\mathfrak{g} \otimes \mathbb{C}$ (or sometimes written with the notation $\mathfrak{g} + i\mathfrak{g}$), with Lie bracket operation given by

  $$[U + iV, X + iY] = [U, X] - [V, Y] + i([V, X] + [U, Y]).$$

- If $T : V \to W$ is a linear map of vector spaces over $\mathbb{R}$, then we can define the **extension**

  $$\overline{T} = T \otimes \mathrm{Id} : V \oplus \mathbb{C} \to W \otimes \mathbb{C}$$

  of T by complex linearity, that is

  $$\overline{T}(\sum v_i \otimes z_i) = \sum T(v_i) \otimes z_i.$$

- Now if $\phi : G \to \mathrm{Aut}(V)$ is a representation of a Lie group G, we combine the previous concepts to define the **complexified** representation

  $$\phi \otimes \mathbb{C} : G \to \mathrm{Aut}(V^{\mathbb{C}}).$$

Example. Let $G = SU(2)$ with the Lie algebra Lie$(SU(2))$ consisting of matrices of the form

$$\begin{pmatrix} is & z \\ -\overline{s} & -i\overline{s} \end{pmatrix}.$$

We will compute the adjoint representation Ad:$SU(2) \to$ Aut(Lie($SU(2)$)). Let

$$A = \begin{pmatrix} x+iy & u+iv \\ -u+iv & x-iy \end{pmatrix} \in SU(2).$$

We know that Ad(A) is a non-singular linear transformation on Lie$SU(2)$ given by Ad$(A)B = ABA^{-1}$.
To find this transformation (actually the matrix that corresponds to this transformation), we pick a basis

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

for Lie$(SU(2))$, and compute

$$\begin{aligned} &\text{Ad}\begin{pmatrix} x+iy & u+iv \\ -u+iv & -x+iy \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ &= \begin{pmatrix} x+iy & u+iv \\ -u+iv & -x+iy \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} x-iy & -u-iv \\ -u-iv & x+iy \end{pmatrix} \\ &= \begin{pmatrix} i(x+iy) & -i(u+iv) \\ -i(u-iv) & -i(x-iy) \end{pmatrix}\begin{pmatrix} x-iy & -(u+iv) \\ u-iv & x+iy \end{pmatrix} \\ &= \begin{pmatrix} ix^2+iy^2-iu^2-iv^2 & 2xv+2uy-2ixu+2ivy \\ -2uy-2xv-2ixu+2ivy & iu^2+iv^2-ix^2-iy^2 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} &\text{Ad}\begin{pmatrix} x+iy & u+iv \\ -u+iv & -x+iy \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x+iy & u+iv \\ -u+iv & -x+iy \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x-iy & -u-iv \\ u-iv & x+iy \end{pmatrix} \\ &= \begin{pmatrix} -(u+iv) & x+iy \\ -(x-iy) & -(u-iv) \end{pmatrix}\begin{pmatrix} x-iy & -(u+iv) \\ u-iv & x+iy \end{pmatrix} \\ &= \begin{pmatrix} -2ixv+2iyu & (u+iv)^2+(x+iy)^2 \\ -(x-iy)^2-(u-iv)^2 & 2ixv-2iyu \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} &\text{Ad}\begin{pmatrix} x+iy & u+iv \\ -u+iv & -x+iy \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} x+iy & u+iv \\ -u+iv & -x+iy \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} x-iy & -(u+iv) \\ u-iv & x+iy \end{pmatrix} \\ &= \begin{pmatrix} i(u+iv) & i(x+iy) \\ i(x-iy) & -i(u-iv) \end{pmatrix}\begin{pmatrix} x-iy & -(u+iv) \\ u-iv & x+iy \end{pmatrix} \\ &= \begin{pmatrix} 2ixu+2iyv & i(x+iy)^2-i(u+iv)^2 \\ i(x-iy)^2-i(u-iv)^2 & -2ixu-2iyv \end{pmatrix}. \end{aligned}$$

Thus

$$\mathrm{Ad}\begin{pmatrix} x+iy & u+iv \\ -u+iv & -x+iy \end{pmatrix}$$
$$= \begin{pmatrix} x^2+y^2-u^2-v^2 & -2xv+2uy & 2xu+2yv \\ 2uy+2xv & x^2-y^2+u^2-v^2 & -2xy+2uv \\ -2xu+2yv & 2xy+2uv & x^2-y^2-u^2+v^2 \end{pmatrix}.$$

This takes a particular simple and useful form on the diagonal element of SU(2):

$$\mathrm{Ad}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}$$

Using the same basis for the Lie algebra, we obtain

$$\mathrm{ad}\begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\theta \\ 0 & 2\theta & 0 \end{pmatrix}.$$

Then a simple calculation using the basis gives that if

$$X = \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} \text{ and } Y = \begin{pmatrix} i\phi & 0 \\ 0 & -i\phi \end{pmatrix},$$

then

$$B(X,Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)) = -8\theta\phi = 4\mathrm{tr}XY.$$