Lie Brackets

- In this section we introduce an important way of combining two smooth vector fields to obtain another veactor field.
- Let V and W be smooth vector fields on a smooth manifolds.
  Given a smooth function $f : M \to \mathbb{R}$, we can apply V to f and obtain another smooth function Vf.
  In turn, we can apply W to this function and obtain another smooth function $WVf = W(Vf)$.

— The operator $f \mapsto WVf$, however, does not in general satisfy the **product rule** and thus cannot be a vector field.

Example. Let $V = \partial/\partial x$ and $W = \partial/\partial y$ on $\mathbb{R}^2$. Let $f(x, y) = x$, $g(x, y) = y$. Then direct computation shows that $VW(fg) = 1$, while $fVWg + gVWf = 0$, so VW is not a derivation of $C^\infty(\mathbb{R}^2)$.

- We can also apply the same two vector fields in the opposite order, obtaining a function WVf.

Definition. *Applying both of these operators to f and subtracting, we obtain an operator $[V, W] : C^\infty(M) \to C^\infty(M)$, called the* **Lie bracket** *of V and W, defined by*

$$[V, W]f = VWf - WVf.$$

- The key fact is that the operation is a vector field.

Lemma 1. *The Lie bracket of any pair of smooth vector fields is a smooth vector field.*

Proof. It suffices to show that $[V, W]$ is a derivation of $C^\infty(M)$. For arbitrary f, $g \in C^\infty(M)$, we compute

$$\begin{aligned}
[V, W](fg) =& V(W(fg)) - W(V(fg)) \\
=& V(fWg + gWf) - W(fVg + gVf) \\
=& VfWg + fVWg + VgWf + gVWf \\
& - WfVg - fWVg - WgVf - gWVf \\
=& fVWg + gVWf - fWVg - gWVf \\
=& f[V, W]g + g[V, W]f. \qquad \square
\end{aligned}$$

Lemma 2. *Let V, W be smooth vector fields on a smooth manifold M, and let $V = V^i \frac{\partial}{\partial x^i}$ and $W = W^j \frac{\partial}{\partial x^j}$ be the coordinate expressions for V and W in terms of some smooth local coordinates (x^i) for M. Then $[V, W]$ has the following coordinate expressions:*

$$[V, W] = \Big(V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i}) \frac{\partial}{\partial x^j}$$

or more precisely,

$$[V, W] = (VW^j - WV^j) \frac{\partial}{\partial x^j}.$$

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Proof. Because we know already that $[V, W]$ is a smooth vector field, its values are determined locally: $([V, W])|_U = [V, W](f|_U)$.

Thus it suffices to compute in a single smooth chart, where we have

$$\begin{aligned}[V, W]f =& V^i \frac{\partial}{\partial x^i}\left(W^j \frac{\partial f}{\partial x^j}\right) - W^j \frac{\partial}{\partial x^j}\left(V^i \frac{\partial f}{\partial x^i}\right) \\ =& V^i \frac{\partial W^j}{\partial x^i}\frac{\partial f}{\partial x^j} + V^i W^j \frac{\partial^2 f}{\partial x^i \partial x^j} - W^j \frac{\partial V^i}{\partial x^j}\frac{\partial f}{\partial x^i} - W^j V^i \frac{\partial^2 f}{\partial x^j \partial x^i} \\ =& V^i \frac{\partial W^j}{\partial x^i}\frac{\partial f}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}\frac{\partial f}{\partial x^i}. \qquad \square\end{aligned}$$

Corollary 3. *For the coordinate vector fields $\frac{\partial}{\partial x^i}$ in any smooth chart, we have*

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0, \quad \forall i, j.$$

Example 4. Define a smooth vector field V, $W \in \mathcal{T}(\mathbb{R}^3)$ by

$$\begin{aligned}V =& \frac{\partial}{\partial z} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z}, \\ W =& \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}.\end{aligned}$$

Then formula (4.6) yields

$$\begin{aligned}[V, W] =& V(1)\frac{\partial}{\partial x} + V(y)\frac{\partial}{\partial z} - W(x)\frac{\partial}{\partial x} - W(1)\frac{\partial}{\partial y} - W(x(y+1))\frac{\partial}{\partial z} \\ =& 0\frac{\partial}{\partial x} + 1\frac{\partial}{\partial z} - 1\frac{\partial}{\partial x} - 0\frac{\partial}{\partial y} - (y+1)\frac{\partial}{\partial z} \\ =& -\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}.\end{aligned}$$

Lemma 5 (Properties of the Lie Bracket). *The Lie bracket satisfies the following identities for all V, W, $X \in \mathcal{T}(M)$:*

(a) **Bilinearity**: $\forall a, b \in \mathbb{R}$,

$$\begin{aligned}[aV + bW, X] =& a[V, X] + b[W, X] \\ [X, aV = bW] =& a[X, V] + b[X, W].\end{aligned}$$

(b) **Antisymmetry**: $[V, W] = -[W, V]$.

(c) **Jacobi identity**:

$$[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0.$$

(d) For f, $g \in C^\infty(M)$,

$$[fV, gW] = fg[V, W] + (fVg)W - (gWf)V.$$

Proof. The proof of the Jacobi identity is just a computation:

$$\begin{aligned}&[V, [W, X]]f + [W, [X, V]]f + [X, [V, W]]f \\ =& V[W, X]f - [W, X]Vf + W[X, V]f - [X, V]Wf + X[V, W]f - [V, W]Xf \\ =& VWXf - VXWf - WXVf + XWVf + WXVf - WVXf \\ &- XVWf + VXWf + XVWf - XWVf - VWXf + WVXf = 0. \quad \square\end{aligned}$$

Proposition 6 (Naturality of the Lie Bracket). *Let $F : M \to N$ be a smooth map, and let V_1, $V_2 \in \mathcal{T}(M)$ and W_1, $W_2 \in \mathcal{T}(N)$ be vector fields such that V_i is F-related to W_i, for $i = 1, 2$. Then $[V_1, V_2]$ is F-related to $[W_1, W_2]$.*

Proof. Using the fact that V_i and W_i are F-related,

$$V_1V_2(f \circ F) = V_1((W_2 f) \circ F) = (W_1W_2f) \circ F.$$

Similarly,

$$V_2V_1(f \circ F) = (W_2W_1f) \circ F.$$

Therefore

$$\begin{aligned}[V_1, V_2](f \circ F) =& V_1V_2(f \circ F) - V_2V_1(f \circ F)\\ =&(W_1W_2f) \circ F - (W_2W_1f) \circ F\\ =&([W_1, W_2]f) \circ F. \qquad \square\end{aligned}$$

Corollary 7. *Suppose $F : M \to N$ is a diffeomorphism and V_1, $V_2 \in \mathcal{T}(M)$. Then $F_*[V_1, V_2] = [F_*V_1, F_*V_2]$.*

Proof. This is just the special case of Proposition 6 in which F is a diffeomorphism and $W_i = F_*V_i$. $\square$