Lie algebras

Definition. *A* **Lie algebra** *is a real vector space* $\mathfrak{g}$ *endowed with a map called the* **bracket** *from* $\mathfrak{g} \times \mathfrak{g}$ *to* $\mathfrak{g}$*, usually denoted by* $(X, Y) \mapsto [X, Y]$*, that satisfies the following properties for all* X, Y, $Z \in \mathfrak{g}$*:*

(1) **Bilinearity***: For* a, $b \in \mathbb{R}$,

$$\begin{aligned}[aX + bY, Z] =& a[X, Z] + b[Y, Z] \\ [Z, aX + bY] =& a[Z, X] + b[Z, Y].\end{aligned}$$

(2) **Antisymmetry***:* $[X, Y] = -[Y, X]$.

(3) **Jacobi's Identity***:*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

- The Jocobi identity is a substitute for **associativity**, which does not hold in general for brackets in a Lie algebra.

Definition. *If* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are Lie algebras, a linear map* $A : \mathfrak{g} \to \mathfrak{h}$ *is called a* **Lie algebra homomorphism** *if it preserves brackets:*

$$A[X, Y] = [AX, AY].$$

- *An invertible Lie algebra homomorphism is called a* **Lie algebra isomorphism**.
- *If there exists a Lie algebra isomorphism from* $\mathfrak{g}$ *to* $\mathfrak{h}$*, we say they are* **isomorphic** *as Lie algebras.*

Examples of Lie Algebras

(a) The space $\Gamma(TM)$ of all smooth vector fields on a smooth manifold M is a Lie algebra under the Lie bracket by Lemma 4.12.

(b) The vector space $M(n, \mathbb{R})$ of $n \times n$ real matrices becomes an n^2-dimensional Lie algebra under the **commutator bracket**

$$[A, B] = AB - BA.$$

— Bilnearity and antisymmetry are obvious from the definition, and the Jacobi identity follows from a straightforward calculation.
— When we are regarding $M(n, \mathbb{R})$ as a Lie algebra with this bracket, we will denote it by $\mathfrak{gl}(n, \mathbb{R})$.

(c) Similarly, $\mathfrak{gl}(n, \mathbb{C})$ is the $2n^2$-dimensional (real) Lie algebra obtained by endowing $M(n, \mathbb{C})$ with the commutator bracket.

(d) If V is a vector space, the linear space $\mathfrak{gl}(V)$ of all linear maps from V to itself becomes a Lie algebra with the commutator bracket:

$$[A, B]x = A(Bx) - B(Ax).$$

— Under our usual identification of $n \times n$ matrices with linear maps from $\mathbb{R}^n$ to itself, $\mathfrak{gl}(\mathbb{R}^n)$ is the same as $\mathfrak{gl}(n, \mathbb{R})$.

(e) Any vector space V becomes a Lie algebra if we define all brackets to be zero.

Definition. *A Lie algebra whose brackets are all zero is said to be* **abelian**.

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The Lie Algebra of a Lie Group

- The most important application of Lie brackets occurs in the context of Lie groups.
- Suppose G is a Lie group. Any $g \in G$ defines maps L_g, $R_g : G \to G$, called **left translation** and **right translation**, respectively, by

$$L_g(h) = gh, \quad R_g(h) = hg.$$

— Because L_g can be written as the composition of smooth maps

$$G \xrightarrow{\imath_g} G \times G \xrightarrow{m} G,$$

where $\imath_g(h) = (g, h)$ and m is multiplication, it follows that L_g is smooth.

— It is actually a diffeomorphism of G, because $L_{g^{-1}}$ is a smooth inverse for it.

⊙ Similarly, $R_g : G \to G$ is a diffeomorphism.

- Observe that, given any two points g_1, $g_2 \in G$, there is a unique left translation of G taking g_1 to g_2, namely left translation by $g_2 g_1^{-1}$.

— Many of the important properties of Lie groups follow from the fact that we can systematically map any point to any other by such a global diffeomorphism.

Definition. *A vector field X on G is said to be* **left-invariant** *if it is invariant under all left translations, in the sense that it is L_g-related to itself for every $g \in G$.*

— *More explicitly, this means*

$$(L_g)_* X_{g'} = X_{gg'}, \quad \forall g, g' \in G. \tag{1}$$

— *Since L_g is a diffeomorphism, this can be abbreviated by writing $(L_g)_* X = X$ for every $g \in G$.*

- Because $(L_g)_*(aX+bY) = a(L_g)_*X + b(L_g)_*Y$, the set of all smooth left-invariant vector fields on G is a linear subspace of $\mathcal{T}(M)$.
- The central fact is that **it is closed under Lie brackets**.

Lemma 1. *Let G be a Lie group, and suppose X and Y are smooth left-invariant vector fields on G. Then $[X, Y]$ is also left-invariant.*

Proof. Since $(L_g)_*X = X$ and $(L_g)_*Y = Y$ by definition of left-invariance, we have

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y].$$

Thus $[X, Y]$ is L_g-related to itself, i.e., is left-invariant. □

- If G is a Lie group, the set of all smooth left-invariant vector fields on G is a Lie subalgebra of $\Gamma(TM)$ and is therefore a Lie algebra. This is called the **Lie algebra of** G, and is denoted by Lie(G), or $\underline{G}$.
- We will see in Corollary 3 below that the assumption of smoothness in (b) is redundant.

- The fundamental fact is that Lie(G) is **finite-dimensional**, and in fact **has the same dimension as G itself**, as the following theorem shows.

Theorem 2. *Let G be a Lie group. The evaluation map $\varepsilon : \mathrm{Lie}(G) \to T_eG$, given by*

$$\varepsilon(X) = X_e,$$

is a vector space isomorphiem. Thus Lie(G) is finite-dimensional, with dimension equal to $\dim G$.

Proof. We will prove the theorem by constructing an inverse for ε.

For each $V \in T_eG$, define a (rough) vector field $\widetilde{V}$ on G by

$$\widetilde{V}_g = (L_g)_* V. \tag{2}$$

If there is a left-invariant vector field on G whose value at the identity e is V, it has to be given by this formula.

(I) First we need to check that $\widetilde{V}$ is **smooth**.

— It suffices to show that $\widetilde{V} f$ is smooth whenever f is a smooth real-valued function on an open set $U \subset G$.

— Choose a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to G$ such that $\gamma(0) = e$, $\gamma'(0) = V$. Then for all $g \in U$,

$$(\widetilde{V} f)(g) = \widetilde{V}_g f = ((L_g)_* V) f = V(f \circ L_g) = \gamma'(0)(f \circ L_g) = \left.\frac{d}{dt}\right|_{t=0} (f \circ L_g \circ \gamma)(t).$$

— If we define $\varphi : (-\varepsilon, \varepsilon) \times G \to \mathbb{R}$ by $\varphi(t, g) = f \circ L_g \circ \gamma(t) = f(g(\gamma(t)))$, the composition above shows that $(\widetilde{V} f)(g) = \frac{\partial \varphi}{\partial t}(0, g)$.

Because φ is a composition of group multiplication, f and γ, it is smooth.

It follows that $\frac{\partial \varphi}{\partial t}(0, g)$ depends smoothly on g, so $\widetilde{V} f$ is smooth.

(II) Next we need to verify that $\widetilde{V}$ **is left-invariant**, which is to say that

$$(L_h)_* \widetilde{V}_g = \widetilde{V}_{hg}, \quad \forall g, h \in G.$$

This follows from the definition of $\widetilde{V}$ and the fact that $L_h \circ L_g = L_{hg}$:

$$(L_h)_* \widetilde{V}_g = (L_h)_* (L_g)_* V = (L_{hg})_* V = \widetilde{V}_{hg}.$$

(III) Finally, we check that **the map $\tau : V \to \widetilde{V}$ is an inverse for ε.**

— On the one hand, given a vector $V \in T_eG$,

$$\varepsilon(\tau(V)) = \varepsilon(\widetilde{V}) = (\widetilde{V})_e = (L_e)_* V = V,$$

which shows that $\varepsilon \circ \tau$ is the identity on T_eG.

— On the other hand, given a vector field $X \in \mathrm{Lie}\,(G)$,

$$\tau(\varepsilon(X))_g = \tau(X_e)_g = \left.\widetilde{X}_e\right|_g = (L_g)_* X_e = X_g,$$

which shows that $\tau \circ \varepsilon = \mathrm{Id}_{\mathrm{Lie}\,(G)}$. ☐

- Given any vector $V \in T_eG$, we will consistently use the notation $\widetilde{V}$ to denote the smooth left-invariant vector field defined by (2).

Corollary 3. *Every left-invariant rough vector field on a Lie group is smooth.*

Proof. Let V be a left-invariant rough vector field on a Lie group G.

The fact that V is left-invariant implies that $V = \widetilde{V}_e$, which is smooth. ☐

Examples. Let us determine the Lie algebra of some familar Lie groups.

(a) **Euclidean space** $\mathbb{R}^n$: Left translation by an element $b \in \mathbb{R}^n$ is given by the affine map

$$L_b(x) = b + x,$$

whose pushforward $(L_b)_*$ is represented by the identity matrix in the standard coordinates.

— Thus a vector field $V^i \frac{\partial}{\partial x^i}$ is left-invariant iff its coefficients V^i are constants.

— Because the Lie bracket of two constant-coefficient vector fields is zero by (4.5), the Lie algebra of $\mathbb{R}^n$ is abelian, and is isomorphic to $\mathbb{R}^n$ itself with the trivial bracket. In brief, $\mathrm{Lie}\,(\mathbb{R}^n) \cong \mathbb{R}^n$.

(b) **The circle group** $\mathbb{S}^1$: In terms of appropraite angle coordinates, each left translation has a coordinate representation of the form

$$\theta \mapsto \theta + c.$$

— Thus the vector $d/d\theta$ is left-invariant, and therefore a basis for the Lie algebra of $\mathbb{S}^1$.

— This Lie algebra is 1-dimensional and abelian, and therefore $\mathrm{Lie}(\mathbb{S}^1) \cong \mathbb{R}$.

(c) **The n-torus** $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$: A similar analysis shows that $(\frac{\partial}{\partial \theta^1}, \cdots, \frac{\partial}{\partial \theta^n})$ is a basis for $\mathrm{Lie}(\mathbb{T}^n)$, where $\frac{\partial}{\partial \theta^i}$ is the angle coordinate vector field on the ith $\mathbb{S}^1$ factor.

— Since the Lie brackets of these coordinate vector fields are all zero, $\mathrm{Lie}(\mathbb{T}^n) \cong \mathbb{R}^n$.

- The Lie groups $\mathbb{R}^n$, $\mathbb{S}^1$, and $\mathbb{T}^n$ are **abelian**, and as the discussion above shows, their Lie algebras turn out to be **abelian**.

— This is no accident:

Theorem. *The Lie algebra of any* **abelian** *Lie group is* **abelian**.

- Just as we can view the tangent space as a "linear model" of a smooth manifold near a point, **the Lie algebra of a Lie group provides a "linear model" of the group**, which reflects many of the properties of the group.
- Because Lie groups have more structure than ordinary smooth manifolds, it should come as no surprise that their linear models have more structure than ordinary vector spaces.
- Since a finite-dimensional Lie algebra is a purely linear-algebraic object, it is **simpler to understand** than the group itself.

— Much of the progress in the theory of Lie groups come from a careful analysis of Lie algebras.

Induced Lie Algebra Homomorphisms

- The importance of the Lie algebra of a Lie group stems, in a large part, from the fact that each Lie group homomorphism induces a Lie algebra homomorphism.

Theorem 4. *Let G and H be Lie groups, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras.*
Suppose $F : G \to H$ is a Lie group homomorphism.
For every $X \in \mathfrak{g}$, there is a unique vector field in $\mathfrak{h}$ that is F-related to X.
With this vector field denoted by F_*X, the map $F_* : \mathfrak{g} \to \mathfrak{h}$ so defined is a Lie algebra homomorphism.

Proof. If there is any vector field $Y \in \mathfrak{h}$ that is F-related to X, it must satisfy $Y_e = F_*X_e$, and thus it must be uniquely determined by

$$Y = \widetilde{F_*X_e}.$$

(i) **To show that Y is F-related to X**, note that the fact that F is a homomorphism implies

$$\begin{aligned} F(gg') = F(g)F(g') &\Longrightarrow F(L_g g') = L_{F(g)}F(g') \\ &\Longrightarrow F \circ L_g = L_{F(g)} \circ F \\ &\Longrightarrow F_* \circ (L_g)_* = (L_{F(g)})_* F_*. \end{aligned}$$

Thus

$$F_*X_g = F_*(L_g)_*X_e = (L_{F(g)})_*F_*X_e = (L_{F(g)})_*Y_e = Y_{F(g)}.$$

This says precisely that X and Y are F-related.

(ii) $\forall X \in \mathfrak{g}$, let F_*X denote the unique vector field in $\mathfrak{h}$ that is F-related to X.
It then follows immediately from the naturality of Lie brackets that

$$F_*[X,Y] = [F_*X, F_*Y],$$

so F_* is a Lie algebra homomorphism. □

Definition. *The map $F_* : \mathfrak{g} \to \mathfrak{h}$ whose existence is asserted in this theorem will be called the* **induced Lie algebra homomorphism**.

- Note that the theorem implies that for any left-invariant vector field $X \in \mathfrak{g}$, F_*X is a well-defined smooth vector field on H, even though F may **not** be a diffeomorphism.

Proposition 5 (Properties of the Induced Homomorphism).

(1) *The homomorphism $(Id_G)_* : Lie(G) \to Lie(G)$ induced by the identity map of G is the identity of Lie(G).*

(2) *If $F_1 : G \to H$ and $F_2 : H \to K$ are Lie group homomorphisms, then*

$$(F_2 \circ F_1)_* = (F_2)_*(F_1)_* : Lie(G) \to Lie(K).$$

(3) *Isomorphic Lie groups have isomorphic Lie algebras.*

Proof. The relations $(\mathrm{Id}_G)_* = \mathrm{Id}$ and $(F_2 \circ F_1)_* = (F_2)_*(F_1)_*$ hold for pushforwards.

— Since the value of the induced homomorphism on a left-invariant vector field X is determined by the pushforward of X_e, this proves (1) and (2).

— If $F : G \to H$ is an isomorphism, (1) and (2) together imply that

$$F_* \circ (F^{-1})_* = (F \circ F^{-1})_* = \mathrm{Id} = (F^{-1})_* \circ F_*,$$

so $F_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is an isomorphism. This proves (3). □