Riemannian Metrics

Symmetric Tensors

Definition. *Let V be a linear algebraic setting. A covaiant k-tensor T on V is said to be* **symmetric** *if its value is unchanged by interchanging any pair of arguments:*

$$T(X_1, \cdots, X_i, \cdots, X_j, \cdots, X_k) = T(X_1, \cdots, X_j, \cdots, X_i, \cdots, X_k),$$

whenever $1 \le i < j \le k$

Definition. *Denote the set of symmetric covariant k-tensors by $S^k(V)$.*

- $S^k(V)$ is obviously a vector subspace of $T^k(V)$.
- There is a natural projection Sym:$T^k(V) \to S^k(V)$ called **symmetrizations**, defined as follows.

Definition. *Let P_k denote the* **symmetric group on k elements**, *i.e. the group of permutation of the sets $\{1, \cdots, k\}$.*
Given a k tensor T and a permutation $\sigma \in P_k$, we define a k-tensor ${}^\sigma T$ by

$${}^\sigma T(X_1, \cdots, X_k) = T(X_{\sigma(1)}, \cdots, X_{\sigma(k)}).$$

Define $\mathrm{Sym}\, T$ by

$$\mathrm{Sym}\, T = \frac{1}{k!} \sum_{\sigma \in P_k} {}^\sigma T.$$

Lemma 1 (Properties of Symmetrization).
(a) For any covariant tensor T, $\mathrm{Sym}\, T$ is symmetric.
(b) T is symmetric iff $\mathrm{Sym}\, T = T$.

Proof. (a) Suppose $T \in T^k(V)$. If $\tau \in S_k$ is any permutation, then

$$(\mathrm{Sym}\, T)(X_{\tau(1)}, \cdots, X_{\tau(k)}) = \frac{1}{k!} \sum_{\eta \in P_k} {}^\eta T(X_{\tau(1)}, \cdots, X_{\tau(k)})$$
$$= \frac{1}{k!} \sum_{\sigma \in P_k} {}^{\sigma\tau} T(X_1, \cdots, X_k) = \frac{1}{k!} \sum_{\eta \in P_k} {}^\eta T(X_1, \cdots, X_k) = \mathrm{Sym}\,(X_1, \cdots, X_k).$$

(b) If T is symmetric, then ${}^\sigma T = T$ $\forall \sigma \in S_k$, and it follows that $\mathrm{Sym}\, T = T$.
On the other hand, if $\mathrm{Sym}\, T = T$, then T is symmetric because part (a) shows that $\mathrm{Sym}\, T$ is. □

- If S and T are symmetric tensors on V, then $S \otimes T$ is not symmetric in general.
- However, using the symmetrization operator, it is possible to obtain a new product that takes symmetric tensors to symmetric tensore.

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Definitin. *If $S \in S^k(V)$ and $T \in S^\ell(V)$, we define the* **symmetric product** *to be the $(k+\ell)$-tensor ST given by*

$$ST = Sym\,(S \otimes T).$$

More explicitly, the action of ST on vectors $X_1, \cdots, X_{k+\ell}$ is given by

$$ST(X_1, \cdots, X_{k+\ell}) = \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell}} S(X_{\sigma(1)}, \cdots, X_{\sigma(k)}) T(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+\ell)}).$$

Lemma 2 (Properties of the Symmetric Product).

(1) *The symmetric product is symmetric and bilinear: For all symmetric tensor R, S, T and a, $b \in \mathbb{R}$,*

$$\begin{aligned} ST =& TS \\ (aR + bS)T =& aRT + bST = T(aR + bS). \end{aligned}$$

(2) *If ω and η are covectors, then*

$$\omega\eta = \frac{1}{2}(\omega \otimes \eta + \eta \otimes \omega).$$

Definition. *A* **symmetric tensor field** *on a manifold is simply a covariant tensor field whose value at any point is a symmetric tensor.*

Riemannian metric

- The most important examples of symmetric tensors on a vector space are inner products.
- Any inner product allows us to define lengths of vectors and angles between them, and thus to do Euclidean geometry.

- Transferring these ideas to manifolds, we obtain one of the most important applications of tensors to differential geometry.

- We now introduce metric structures on differentiable manifolds.

— We shall start from infinitesimal considerations.

— We want to be able to measure the lengths and the angles between tangent vectors. Then, one may obtain the length of a differentiable curve by integratiion.

— In a vector space such a notion of mesurement is usually given by a scalar product.

- A Riemannian metric on an open set U of $\mathbb{R}^n$ is a family of positive definite quadratic forms on $\mathbb{R}^n$, depending smoothly on $p \in U$.

Definition 2.1*. *A* **Riemannian metric** *on a differentiable mfd M is given by a scalar product on each tangent space T_pM which depends smoothly on the base point p.*

- *A* **Riemannian mfd** *is a differentiable mfd equipped with a Riemannian metric.*

Definition 2.1. *A* **Riemannian metric** *on a smooth manifold M is a $\binom{2}{0}$-tensor field $g \in \mathcal{T}^2(M)$ that is*

(1) **symmetric** *(i.e. $g(X, Y) = g(Y, X)$, $\forall X, Y \in T_pM$, $p \in M$), and*
(2) **positive definite** *(i.e. $g(X, X) > 0$, if $X \neq 0$).*

- A Riemannian metric thus determines an inner product on each tangent space T_pM, which typically written

$$\langle X, Y \rangle = g(X, Y), \quad \forall X, Y \in T_pM.$$

Definition. *A* **Riemannian manifold** *is a pair (M, g), where M is a smooth manifold and g is a Riemannian metric on M.*

- Note that a Riemannian metric is not the same thing as a metric in the sense of metric spaces, although the two concepts are closely related.

— Because of this ambiguity, we will usually

(1) use the term "distance function" when considering a metric in the metric space sense, and
(2) reserve "metric" for a Riemannian metric.

- In any smooth coordinate coordinates (x^i), a Riemannian metric can be written

$$g = \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j,$$

  where g_{ij} is a symmetric positive definite matrix of smooth functions
  (i.e. $g_{ij} = g_{ji}$ $\forall i, j$, and $g_{ij}\xi^i\xi^j > 0$, $\forall \xi = (\xi^1, \cdots, \xi^n) \neq 0$),
  where the coefficients depend smoothly on x such that

$$(g_{ij}(x))_{i,j=1,\cdots,n} = \langle \frac{\partial}{\partial x^i}\Big|_x, \frac{\partial}{\partial x^j}\Big|_x \rangle = g(\frac{\partial}{\partial x^i}\Big|_x, \frac{\partial}{\partial x^j}\Big|_x).$$

- Observe that the symmetry of g allows us to write g also in terms of symmetric products as follows:

$$\begin{aligned} g &= \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j \\ &= \frac{1}{2} \sum_{i,j=1}^{n} (g_{ij} dx^i \otimes dx^j + g_{ji} dx^i \otimes dx^j) \quad (\because \quad g_{ij} = g_{ji}) \\ &= \frac{1}{2} \sum_{i,j=1}^{n} (g_{ij} dx^i \otimes dx^j + g_{ij} dx^j \otimes dx^i) \\ &= \sum_{i,j=1}^{n} g_{ij} dx^i dx^j. \end{aligned}$$

- The product of two tangent vectors u, $v \in T_pM$ with coordinate representations $(u^1, \cdots, u^n)$ and $(v^1, \cdots, v^n)$, (i.e. $u = u^i \frac{\partial}{\partial x^i}$, and $v = v^j \frac{\partial}{\partial x^j}$) then is

$$\langle u, v \rangle = g_{ij}(x(p)) u^i v^j = g_m(u, v).$$

- The length of v is given by $\|v\| = \langle v, v \rangle^{1/2}$.

Example. The simplest example of a Riemannian metric is the **Euclidean metric** $\widetilde{g}$ on $\mathbb{R}^n$, defined in standard coordinates by

$$\overline{g} = \delta_{ij} dx^i dx^j = (dx^1)^2 + \cdots + (dx^n)^n,$$

where δ_{ij} is the Kronecker delta. Applying to vectors v, $w \in T_p\mathbb{R}^n$, this yield

$$\overline{g}_p(v, w) = \delta_{ij} v^i w^j = \sum_{i=1}^{n} v^i w^i = v \cdot w.$$

In other words, $\overline{g}$ is the 2-tensor field whose value at each point is the Euclidean product.

Existence of Riemannian Metrics

Theorem 2.2. *Every smooth manifold admits a Riemannian metric.*

Proof 1. Begin by covering M by smooth coordinate charts $(U_\alpha, \varphi_\alpha)$.

— In each coordinate domain, there is a Riemannian metric g_α given by the Euclidean metric $\overline{g} = g\delta_{ij}dx^i dx^j$ in coordinates; that is,

$$g_\alpha(X, Y) = \overline{g}(\varphi_* X, \varphi_* Y).$$

— Let $\{\psi_\alpha\}$ be a smooth partition of unity subordinate to the cover $\{U_\alpha\}$, (cf. Theorem 3 below), and define

$$g = \sum_\alpha \psi_\alpha g_\alpha.$$

– Because of the **local finiteness** condition for partitions of unity, there are only finitely many nonzero terms in a nbhd of any point, so the expression defines a smooth tensor field.

(i) It is obviously symmetric.

(ii) We only need to check the **positivity** of g. If $X \in T_pM$ is any nonzero vector, then

$$g_p(X, X) = \sum_\alpha \psi_\alpha(p) g_\alpha\Big|_p(X, X).$$

This sum is nonnegative, because each term is nonnegative.
At least one of the function ψ_α is strictly positive at p (because they sum to 1).
Thus $g_\alpha\big|_p(X, X) > 0$, and hence $g_p(X, X) > 0$. □

Proof 2. The second proof relies on the Whitney embedding theorem (cf. Theorem 4 below). We simply embed M in $\mathbb{R}^N$ for some N, and then the Euclidean metric induces a Riemannian metric $g\big|_M$ on M. □

Definition. *Let X be a topological apace. A collection $\mathcal{U}$ of subsets of X is said to be* **locally finite** *if each point of X has a nbhd that intersects at most finitely many of the sets in $\mathcal{U}$.*

Definition. *Let M be a topological space, and let $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ be an arbitrary open cover of M. A* **partition of unity subordinate to** *$\mathcal{X}$ is a collection of functions $\{\psi_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ with the following properties*

(i) $0 \le \psi_\alpha(x) \le 1$ *for all* $\alpha \in A$ *and* $x \in M$.

(ii) $\operatorname{supp} \psi_\alpha \subset X_\alpha$.

(iii) The set of supports $\{\operatorname{supp} \psi_\alpha\}_{\alpha \in A}$ *is locally finite.*

(iv) $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ *for all* $x \in M$.

- Because of the local finiteness (iii), the sum in (iv) actually has only finitely many nonzero terms in a nbhd of each point, so there is no issue of convegence.

Theorem 3 (Existence of Partition of unity). *Let M be a topological space, and let $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ be an arbitrary open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.*

Theorem 4 (Whitney Embedding Theorem). *Every smooth n-manifold admits a proper embedding into $\mathbb{R}^{2n+1}$.*

- Below are a few geometric constructions that can be defined on a Riemannian manifold (M, g).
  (1) The **length** or **norm** of a tangent vector $X \in T_pM$ is defined to be
  $$|X|_q = \langle X, X\rangle^{1/2} = g_p(X, X)^{1/2}.$$
  (2) The **angle** between two nonzero tangent vectors $X, Y \in T_pM$ is the unique $\theta \in [0, \pi]$ satisfying
  $$\cos\theta = \frac{\langle X, Y\rangle_g}{|X|_g |Y|_g}.$$
  (3) Two tangent vectors $X, Y \in T_pM$ are said to be orthogonal if $\langle X, Y\rangle_g = 0$.

Psudo-Riemannian metric

Definition. *A $\binom{2}{0}$-tensor g on a vector space V is said to be* **nondegenerate**
$\Leftrightarrow$ *$g(X, Y) = 0$ for all $Y \in V$ iff $X = 0$;*
$\Leftrightarrow$*The only vector orthogonal to every vector is the zero vector.*
$\Leftrightarrow$*The matrix (g_{ij}) is invertible if $g = g_{ij}\varepsilon^i\varepsilon^j$ in terms of a local coframe $\{\varepsilon^i\}$.*

- Every nondegenerate symmetric $\binom{2}{0}$-tensor can be transformed by a change of basis to one whose matrix is diagonal with all entries equal to ± 1; i.e. one can construct a basis $(E_1, \cdots, E_n)$ for T_pM in which g has the expression
$$g = -(\varepsilon^1)^2 - \cdots - (\varepsilon^r)^2 + (\varepsilon^{r+1})^2 + \cdots + (\varepsilon^n)^2$$

— The integer r, called the **index** of g, is equal to the maximum dimension of any subspaace of T_pM on which g is negative definite.
— Therefore, the index is independent of the choice of basis.

Definition. *The* **signature** *of g is the sequence $(-1, \cdots, -1, 1, \cdots, 1)$ of diagonal entries in nonincreasing order.*

- The signature is an invariant of g.

Definition. *A* **psudo-Riemannian metric** *on a manifold M is a smooth symmetric $\binom{2}{0}$-tensor field whose value is nondegenerate at each point.*

Definition. **Lorentz metrics** *are Psudo-Riemannian metrics with signature*
$$(-1, +1, \cdots, +1).$$

Minkowski metric *is the Lorentz metric m on $\mathbb{R}^{n+1}$ that is written in terms of coordinates $(\xi^1, \cdots, \xi^n, \tau)$ as*
$$m = (d\xi^1)^2 + \cdots + (d\xi^n)^2 - (d\tau)^2.$$

Remark. **Neither** of the proofs we gave of the existence of Riemannian metrics carries over to the pseudo-Riemannian case. In particular,
(1) it is not alwats true that the restriction of a nonnegative 2-tensor to a subspace is nonndegenerate,
(2) nor is it true that a linear combination of nondegenerate 2-tensor with positive coefficients is necessarily nondegenerate.

Indeed, it is not true that every manifold admits a Lorentz metric.

Isometry

Definition. *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds.*

- *A smooth map $F : M \to \widetilde{M}$ is called an* **isometry** *if it is a diffeomorphism that satisfies $F^*\widetilde{g} = g$.*

— *If there exists an isometry between M and $\widetilde{M}$, we say that M and $\widetilde{M}$ are* **isometric** *as Riemannian manifolds.*

- *F is called a* **local isometry** *if every point $p \in M$ has a nbhd U such that $F\big|_U$ is an isometry of U onto an open subset of $\widetilde{M}$.*

- **Riemannian geometry** is the study of properties of Riemannian manifold that are invariant under isometries.

Definition. *A metric g on M is said to be* **flat** *if every point $p \in M$ has a nbhd $U \subset M$ such that $(U, g\big|_U)$ is isometric to an open subset of $\mathbb{R}^n$ with the Euclidean metric.*

Orthonormal frames

- Another extremely useful tool on Riemannian mfds is orthonormal frames.

Definition. *Let (M, g) be an n-dimensional Riemannian manifold. An* **orthonormal frame** *for M is a local frame $(E_1, \cdots, E_n)$ defined on some open subset $U \subset M$ such that $(E_1|_p, \cdots, E_n|_p)$ is an orthonormal basis at each point $p \in U$, or equivalently such that $\langle E_i, E_j \rangle_g = \delta_{ij}$.*

Example. The coordinate frame $(\partial/\partial x^i)$ is a global orthonormal frame on $\mathbb{R}^n$.

Proposition 2 (Existence of Orthonormal Frames). *Let (M, g) be a Riemannian manifold. $\forall p \in M$, $\exists$a smooth orthonormal frame on a nbhd of p.*

Proof. Let (x^i) be any smooth coordinates on a nbhd U of p,
and apply the Gram-Schmidt algorithm to the coordinate frame $(\partial/\partial x^i)$.
This yields a smooth orthonormal frame on U. $\square$

- Observe that Proposition 2 does **not** show that there are smooth coordinates near p for which the **coordinate frame** is orthonormal.

Proposition. *The following are equivalent:*

(1) *Each point of M has a smooth coordinate nbhd in which the coordinate frame is orthonormal.*
(2) *g is flat.*

Length and Distances on Riemannian Manifolds

- We are now in a position to introduce two of the most fundamental concepts from classical the Riemannian geoometry into the Riemannian setting: **length of curves** and **distances between points**.

Length of Curves

Definition. *If $\gamma : [a,b] \to M$ is a piecewise smooth curve segment, we define the* **length of** γ *to be*

$$L_g(\gamma) = \int_a^b |\gamma'(t)|_g \, dt.$$

- Because $|\gamma'(t)|_g$ is continuous at all but finitely many values of t, and has well-defined left- and right-handed limits at those points, the length is well-defined.

The Riemannian Distance Function

- Using curve segments as "measuring tapes", we can define a notion of distance between points on a Riemannian manifold.

Definition. *If (M, g) is a connected Riemannian manifold and p, $q \in M$, the* **(Riemannian) distance** *between p and q, denoted by $d_q(p, q)$, is defined to be the infimum of $L_g(\gamma)$ over all piecewise smooth curve segments γ from p to q.*

- Because any pair of points in a connected smooth manifold can be joined by a piecewise smooth curve segment, this is well-defined.

Example. On $\mathbb{R}^n$ with the Euclidean metric $\overline{g}$, one can show that any straight line segment is the shortest piecewise smooth curve segment between its endpoints. Therefore, the distance function $d_{\overline{g}}$ is equal to the usual Euclidean distance:

$$d_{\overline{g}}(x, y) = |x - y|.$$

- We will show below the following.

Theorem. *The Riemannian distance function turn M into a metric space whose topology is the same as the given manifold topology.*

- **Transformation Behavior under Coordinate Change:**

  Let $y = f(x)$ define different local coordinates around m, v and w have representations $(\tilde{v}^1, \cdots, \tilde{v}^d)$ and $(\tilde{w}^1, \cdots, \tilde{w}^d)$ with

$$\tilde{v}^j = v^i \frac{\partial f^j}{\partial x^i}, \quad \tilde{w}^j = w^i \frac{\partial f^j}{\partial x^i}.$$

  Let the metric in the new coordinates be given by $h_{k\ell}(y)$. Then

$$h_{k\ell}(f(x))\tilde{v}^k \tilde{w}^\ell = \langle v, w \rangle = g_{ij}(x) v^i w^j;$$

  hence

$$h_{k\ell}(f(x)) \frac{\partial f^k}{\partial x^i} \frac{\partial f^\ell}{\partial x^j} v^i w^j = g_{ij}(x) v^i w^j,$$

  and since this holds for all tangent vectors v, w,

$$h_{k\ell} \frac{\partial f^k}{\partial x^i} \frac{\partial f^\ell}{\partial x^j} = g_{ij}(x). \tag{1.4.3}$$