Length and Distances on Riemannian Manifolds

- We are now in a position to introduce two of the most fundamental concepts from classical the Riemannian geoometry into the Riemannian setting: **length of curves** and **distances between points**.

Length of Curves

Definition. *If $\gamma : [a, b] \to M$ is a piecewise smooth curve segment, we define the* **length of** γ *to be*

$$L_g(\gamma) = \int_a^b |\gamma'(t)|_g \, dt.$$

- Because $|\gamma'(t)|_g$ is continuous at all but finitely many values of t, and has well-defined left- and right-handed limits at those points, the length is well-defined.
- The key feature of the length of a curve is that it is **independent of the parametrization**.

Definition. *A* **reparametrization** *of a piecewise smooth segment $\gamma : [a, b] \to M$ is a curve segment of the form $\widetilde{\gamma} = \gamma \circ \varphi$, where $\varphi : [c, d] \to [a, b]$ is a diffeomorphism.*

Proposition 1 (Parameter Independence of Length). *Let (M, g) be a Riemannian manifold, and let $\gamma : [a, b] \to M$ be a piecewise smooth curve segment. If $\widetilde{\gamma}$ is any reparametrization of γ, then $L_g(\widetilde{\gamma}) = L_g(\gamma)$.*

Proof. (I) First suppose that γ is smooth. Let $\varphi : [c, d] \to [a, b]$ be a diffeomorphism such that $\widetilde{\gamma} = \gamma \circ \varphi$. The fact that φ is a diffeomorphism implies that either $\varphi' > 0$ or $\varphi' < 0$.

(i) Let us assume first that $\varphi' > 0$. We have

$$\begin{aligned} L_g(\widetilde{\gamma}) &= \int_c^d |\widetilde{\gamma}'(t)|_g dt = \int_c^d |\frac{d}{dt}(\gamma \circ \varphi)(t)|_g dt \\ &= \int_c^d |\varphi'(t)\gamma'(\varphi(t))|_g dt = \int_c^d |\gamma'(\varphi(t))|_g \varphi'(t) dt \\ &= \int_a^b |\gamma'(s)|_g ds = L_g(\gamma). \end{aligned}$$

(ii) In case $\varphi' < 0$, we need to interduce two sign changes into the above calculations.

   (1) The sign changes once when $\varphi'(t)$ is moved outside the absolute value signs, because $|\varphi'(t)| = -\varphi'(t)$.

   (2) Then it changes again when we change varable, because φ reverses the direction of the integral.

(II) If γ is only piecewise smooth, we apply the same argument on each subinterval on which it is smooth. $\square$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

The Riemannian Distance Function

- Using curve segments as "measuring tapes", we can define a notion of distance between points on a Riemannian manifold.

Definition. *If (M, g) is a connected Riemannian manifold and p, $q \in M$, the* **(Riemannian) distance** *between p and q, denoted by $d_q(p, q)$, is defined to be the infimum of $L_g(\gamma)$ over all piecewise smooth curve segments γ from p to q.*

- Because any pair of points in a connected smooth manifold can be joined by a piecewise smooth curve segment, this is well-defined.

Example. On $\mathbb{R}^n$ with the Euclidean metric $\overline{g}$, one can show that any straight line segment is the shortest piecewise smooth curve segment between its endpoints. Therefore, the distance function $d_{\overline{g}}$ is equal to the usual Euclidean distance:

$$d_{\overline{g}}(x, y) = |x - y|.$$

- We will show below that the Riemannian distance function turn M into a metric space whose topology is the same as the given manifold topology.

— The key is the following technical lemma, which shows that any Riemannian metric is locally comparable to the Euclidean metric in coordinates.

Lemma 2. *Let (M, g) be any Riemannian manifold of dimension n. Let $\overline{g}$ be the Euclidean metric on $\mathbb{R}^n$. For any coordinate chart (U, φ) around $p \in M$ and any compact set $K \subset U$, there exist positive constants c, C such that $\forall x \in K$ and $\forall v \in T_xM$,*

$$(1) \qquad c|v|_g \leq |\varphi_* v|_{\overline{g}} \leq C|v|_g, \quad \text{where } |v|_g = (g(v, v))^{1/2} \text{ and } |v|_{\overline{g}} = (\overline{g}(v, v))^{1/2}.$$

Proof. For any compact set $K \subset U$, let $L \subset TM$ be the set

$$L = \{(x, v) \in TM : \; x \in K, \; |v|_g = 1\}.$$

L is a compact subset of TM.

Since the norm $|\varphi_* v|_{\overline{g}}$ is **continuous** and strictly positive, $\exists$positive constants c, C such that

$$c \leq |\varphi_* v|_{\overline{g}} \leq C \quad \text{whenever } (x, v) \in L.$$

(a) If $s \in K$ and v is any nonzero vector in T_xM, let $\lambda = |v|_g$.
(i)Then $(x, \lambda^{-1}v) \in L$, so by homogeneity of the norm,

$$|\varphi_* v|_{\overline{g}} = \lambda|\lambda^{-1}v|_{\overline{g}} \leq \lambda C = C|v|_g.$$

(ii)A similar computation shows that $|\varphi_* v|_{\overline{g}} \geq c|v|_g$.

(b) The same inequalities are trivially true when $v = 0$. $\square$

Lemma 3. *Let* (M, g) *be any Riemannian manifold of dimension* n. *Let* $\overline{g}$ *be the Euclidean metric on* $\mathbb{R}^n$. *For any coordinate chart* (U, φ) *around* $p \in M$, *let* $\rho > 0$ *be so small that* $\overline{B_\rho(\varphi(p))} \subset \varphi(U)$, *and* $\overline{\varphi^{-1}(B_\varepsilon(\varphi(p)))} \subset U$. *Then there exist positive constants* c, C *such that* $\forall q \in \varphi^{-1}(B_\rho(\varphi(p)))$,

$$cd(p,q) \leq d'(\varphi(p), \varphi(q)) \leq Cd(p,q), \tag{2}$$

where $d'(\varphi(p), \varphi(q))$ *is the Euclidean distance from* $\varphi(p)$ *to* $\varphi(q)$.

Proof. Let $V = \varphi^{-1}(B_\rho(\varphi(p)))$. Since $\overline{V}$ is a compact subset of U, Lemma 2 shows that $\exists$positve constants c, C such that

$$c|X|_g \leq |\varphi_* X|_{\overline{g}} \leq C|X|_g, \quad \text{whenever } x \in \overline{V} \text{ and } X \in T_x M. \tag{**}$$

Then for any piecewise smooth curve segment γ lying in $\overline{V}$, it follows that

$$cL_g(\gamma) \leq L_{\overline{g}}(\gamma \circ \varphi) \leq CL_g(\gamma).$$

Fix $q \in V$.

(I) Suppose $\gamma : [a, b] \to M$ is an arbitrary curve with $\gamma(a) = p$ and $\gamma(b) = q$.

(i) If $\gamma([a, b]) \subset \overline{V}$, then

$$d'(\varphi(p), \varphi(q)) \leq L_{\overline{g}}(\varphi \circ \gamma) \leq CL(\gamma).$$

(ii) If $\gamma([a, b])$ does not lie entirely in $\overline{V}$, define

$$t = \max\{s : \gamma(\lambda) \subset \overline{V}, \ \forall \lambda \leq s\}.$$

Then $t < b$. Also

$$d'(\varphi(p) - \varphi(\gamma(t))) = \rho$$

and

$$d'(\varphi(p), \varphi(q)) \leq \rho = L_{\overline{g}}((\varphi \circ \gamma)\Big|_{[a,t]}) \leq CL_g(\gamma\Big|_{[a,t]}) \leq CL_g(\gamma).$$

So in either case, $d'(\varphi(p), \varphi(q)) \leq CL(\gamma)$. Taking the infimum over all such γ, we find that

$$d'(\varphi(p), \varphi(q)) \leq Cd(p, q).$$

(II) On the other hand, let $\varphi(\gamma)$ be the straight-line segment from $\varphi(p)$ and $\varphi(q)$; i.e.

$$\gamma(t) = \varphi^{-1}(\varphi(p) + t(\varphi(q) - \varphi(p))).$$

Then $\gamma(t)$ has image in V and so has length $L_g(\gamma)$ satisfying

$$cd(p, q) \leq cL_g(\gamma) \leq L_{\overline{g}}(\varphi \circ \gamma) = d'(\varphi(p), \varphi(q)). \qquad \square$$

Proposition 2.91 (Riemannian Manifolds as Metric Spaces). *Let (M, g) be a connected Riemannian manifold. With the Riemannian distance function, M is a metric space whose metric topology is the same as the original manifold topology.*

Proof. **(I) Claim: (M, d_g) is a metric space.**

(a) It is immediate from the definition that $d_g(p, q) \geq 0$ for all p, $q \in M$.

(b) Because any constant curve segment has length zero, it follows that $d_g(p, p) = 0$.

(c) $d_g(p, q) = d_g(q, p)$, since any curve segment from p to q can be reparametrized to go from q to p.

(d) Suppose γ_1 and γ_2 are piecewise smooth curve segments from p to q and q to r, respectively. Let γ be a piecewise smooth curve segment that first follow γ_1 and then follow γ_2 (reparametrized if necessary). Then

$$d_g(p, r) \leq L_g(\gamma) = L_g(\gamma_1) + L_g(\gamma_2).$$

Taking the infimum over all such γ_1 and γ_1, we find that

$$d_g(p, r) \leq d_g(p, q) + d_g(q, r).$$

(This is one reason why it is important to define the distance function **using piecewise smooth curves instead of only smooth ones.**)

(e) **Claim: $d_g(p, q) > 0$ if $p \neq q$.**

— Let (U, p) be any smooth coordinate chart with $p \in U$ but $q \notin U$.

— Let $V = \varphi^{-1}(B_\varepsilon(\varphi(p)))$ such that $\overline{V} \subset U$. **Lemma 3** shows that ∃positve constant C such that $d_g(p, q) \geq C^{-1}\varepsilon > 0$.

(II) To show that **the metric topology generated by d_g** is the same as **the given manifold topology on M**, we will show that **the open set in the manifold are open in the metric topology and vice versa**.

(i) Suppose first that $U \subset M$ is open in the manifold topology.

— Let p be any point of U, and let $V = \varphi^{-1}(B_\varepsilon(\varphi(p)))$ such that $\overline{V} \subset U$.

— The argument in (I)(e) shows that

$$d_g(p, q) \geq \varepsilon / C \quad \text{whenever } q \notin \overline{V}.$$

— The contrapositive of this statement is that

$$d_g(p, q) < \varepsilon / C \Rightarrow q \in \overline{V} \subset U,$$

or in other words, the metric ball of radius ε / C around p is contained in U.

— This shows that U is open in the metric topology.

(ii) Conversely, suppose that W is open in metric topology.

— Choosing $p \in W$, let V be any smooth coordinate ball $V = \varphi^{-1}(B_\varepsilon(\varphi(p)))$ around p, and let c, C be positive constants such that (2) is satisfied for $q \in \overline{V}$.

— Choose ε small enough that the closed metric ball $\overline{B_{\varepsilon/c}(p)}$ is in W.

— Let $V_\varepsilon = \{q \in M : \ d'(\varphi(p), \varphi(q)) \leq \varepsilon\}$. Lemma 3 shows that $V_\varepsilon \subseteq \overline{B_{\varepsilon/c}(p)} \subset W$.

— Since V_ε is a nbhd of p in the manifold topology, this shows that W is open in the manifold topology. □