Jacobi Fields

Definition. *Any vector field satisfied* **the Jacobi equation**

$$D_t^2 V + R(\gamma', V)\gamma' = 0.$$

along a geodesic is called a **Jacobi field**.

- Suppose therefore that $\gamma : [a, b] \to M$ is a geodesic segment, and $H : (-\varepsilon, \varepsilon) \times [a, b] \to M$ is a variation of γ.

Definition. *We say H is a* **variation through geodesics** *if each of the main curves $H_s(t) = H(s, t)$ is also a geodesic segment.*
(In particular, this requires that H be smooth.)

- Write $T(s, t) = \partial_t H(s, t)$ and $S(s, t) = \partial_s H(s, t)$. The geodesic equation says that
$$D_t T \equiv 0, \quad \forall (s, t).$$

We can take the covariant derivative of this equation with respect to s, yielding

$$D_s D_t T \equiv 0.$$

- To relate this to the variation field of γ, we need to commute the covariant differentiation operators D_s and D_t.
- Because these are covariant derivatives acting on a vector field along a curve, we should expect the curvature to be involved.

Theorem 1. *Let γ be a geodesic and V a vector field along γ. If V is the variational field of a variation through geodesics, then V satisfies the Jacobi equation.*

Proof. With S and T as before, the preceeding lemma implies

$$\begin{aligned}
0 = D_s D_t T &= D_t D_s T + R(T, S)T \\
&= D_t D_t S + R(T, S)T,
\end{aligned}$$

where the last step follows from the symmetry lemma.
Evaluating at $s = 0$, where $S(0, t) = V(t)$ and $T(0, t) = \gamma'(t)$, we obtain (7). $\square$

- The following lemma is a converse to Theorem 1: each Jacobi field tells us how some family of geodesics behaves, at least "infinitesimally" along γ.

Lemma 2. *Every Jacobi field along a geodesic γ is the variation field of some variation of γ through geodesics.*

- Now we reverse our approach: let us forget about variations for a while, and just study Jacobi fields in their own right.
— As the following proposition shows, the Jacobi equation can be written as a system of second order linear ODEs, so it has a unique solution given initial values for V and $D_t V$ at one point.

Proposition 3 (Existence and Uniqueness of Jacobi Fields). *Let $\gamma : I \to M$ be a geodesic, $a \in I$ and $p = \gamma(a)$. For any pairs of vectors $X, Y \in T_pM$, there is a unique Jacobi field J along γ satisfying the the initial conditions*

$$J(a) = X; \quad D_t J(a) = Y.$$

Proof. Choose an orthonormal basis $\{E_i\}$ for T_pM, and extend it to a parallel orthonormal frame along all of γ. Then $J(t)$ can me wriien as

$$J(t) = J^i(t) E_i(t).$$

Setting

$$R_{jk\ell}{}^i(t) = \langle R(E_j, E_k)E_\ell, E_k \rangle(t);$$

we can express the Jacobi equation as

$$(J^i)'' + R_{jk\ell}{}^i J^k (\gamma^j)'(\gamma^\ell)' = 0.$$

This is a **linear** system of second order ODEs for the n functions J^i.

— Making the usual substitution $V^i = \dot{J}^i$ converts it to an equivalent first-order linear system for the $2n$ unknowns $\{J^i, V^i\}$.

— This guarantees the existence and uniqueness of a solution on the whole interval I with any initial conditions $J^i(a) = X^i$, $V^i(a) = Y^i$. $\quad \square$

Corollary 4. *Along any geodesic γ, the set of Jacobi fields is a $2n$-dimensional linear subspace of $\mathcal{T}(\gamma)$.*

Proof. Let $p = \gamma(a)$ be any point of γ, and consider the map from the set of Jacobi fields along γ to $T_pM \oplus T_pM$ by sending J to $(J(a), D_t J(a))$.

The preceeding proposition says precisely that this map is linear. $\quad \square$

• There are always two trivial Jacobi fields along any geodesic.

(1) Because $D_t\gamma' = 0$ and $R(\gamma', \gamma')\gamma' = 0$ by antisymmetry of R, the vector field $J_0(t) = \gamma'(t)$ satisfies the Jacobi equation with initial conditions

$$J_0(0) = \gamma'(0); \quad D_t J_0(0) = 0.$$

(2) Similarly, $J_1(t) = t\gamma'(t)$ is a Jacobi field with initial conditions

$$J_1(0) = 0; \quad D_t J_1(0) = \gamma'(0).$$

— J_0 is the variation field of the variation $H(s,t) = \gamma(s+t)$, while J_1 is the variation field of $H(s,t) = \gamma(e^s t)$.

— Therefore, these two Jacobi fields just reflect the possible reparametrizations of γ, and do not tell us anything about the behavior of geodesics other than γ itself.

Definition. *A **tangential vector field** along a curve γ is a vector field V such that $V(t)$ is a multiple of $\gamma'(t)$, $\forall t$.*
*A **normal vector field** along a curve γ is a vector field V s.t. $V(t) \perp \gamma'(t)$, $\forall t$.*

Lemma 5. *Let $\gamma : I \to M$ be a geodesic and $a \in I$.*
(a) A Jacobi field along γ is normal iff

$$J(a) \perp \gamma'(a) \;\; and \;\; D_t J(a) \perp \gamma'(a).$$

(b) Any Jacobi field orthogonal to γ' at two points is normal.

Proof. Using compatibility with the metric and the fact that $D_t \gamma' \equiv 0$, we compute

$$\frac{d^2}{dt^2} \langle J, \gamma' \rangle = \langle D_t^2 J, \gamma' \rangle = -\langle R(J, \gamma')\gamma', \gamma' \rangle = -Rm(J, \gamma', \gamma', \gamma') = 0$$

by the symmetries of the curvature tensor. Therefore

$$f(t) := \langle J(t), \gamma'(t) \rangle$$

is a linear function of t.
(a) Note that
$$f(a) = \langle J(a), \gamma'(a) \rangle \;\; and \;\; f'(a) = \langle D_t J(a), \gamma'(a) \rangle.$$

Thus $J(a)$ and $D_t J(a)$ are orthogonal to $\gamma'(a)$ iff f and its first derivative vanish at a, which happens iff $f \equiv 0$.
(b) Similarly, if J is orthogonal to γ' at two points, then f vanishes at two points and is therefore identically zero. $\square$

- As a consequence of this lemma, it is easy to check that the space of **normal Jacobi fields** is $(2n-2)$-dimensional subspace of $\mathcal{T}(\gamma)$, and the space of **tangential** ones is a 2-dimensional subspace.
- Every Jacobi field can be uniquely decomposed into the sum of a tangeial Jacobi field plus a normal Jacobi field, **just by decomposing its initial value and initial derivative**.

- In Riemannian normal coordinates, half of the Jacobi fields are easy to write down explicitly.

Lemma 6. *Let $p \in M$, let (x^i) be normal coordinates on a nbhd $\mathcal{U}$ of p, and let γ be a radial geodesic starting at p. For any $W = W^i \partial_i \in T_p M$, the Jacobi field J along γ such that $J(0) = 0$ and $D_t J(0) = W$ is given in normal coordinates by*

(*) $$J(t) = t W^i \partial_i.$$

Proof. It is easy to check that J satisfies the specified initial conditions.
⊙ Thus it suffices to **claim: J is a Jacobi field**.
— If we set $V = \gamma'(0) \in T_p M$, then γ is given in coordinates by the formula

$$\gamma(t) = (tV^1, \cdots, tV^n).$$

Now consider the variation H given in coordinates by

$$H(s, t) = (t(V^1 + sW^1), \cdots, t(V^n + sW^n)).$$

We see that H is a variation through geodesics.
– Therefore its variation field $\partial_s H(0, t)$ is a Jacobi field.
– Differentiating $H(s, t)$ with respect to s shows that its variation field is $J(t)$. $\square$

Jacobi Fields and the Exponential Map

Question: When is the exponential map a local diffeomorphism?

- If (M, g) is complete, we know that $\exp_p$ is defined on all of T_pM, and is a local diffeomorphism near 0.
— However, it may well happen that it ceases to be even a local diffeomorphism at points far away.

Example. An enlightening example is provided by the sphere $\mathbb{S}^n_R$.
— All geodesics starting at a given point p meet at the antipodal point, which is at a distance of πR along each geodesic.
— The exponential map is a diffeomorphism on the ball $B_{\pi R}(0)$, but it fails to be a local diffeomorphism at all points on the sphere of radius πR in $T_p\mathbb{S}^n_R$.
— Moreover, each Jacobi field on $\mathbb{S}^n_R$ vanishes at p has its first zero precisely at distance πR.

- Formula (*) shows that if $\mathcal{U}$ is a normal nbhd of p (the image of a set on which $\exp_p$ is a diffeomorphism), no Jacobi field that vanishes at p can vanish at any other point.
— We might thus to expect a relationship between zeros of Jacobi fields and singularities of the exponential map (i.e. points where it fails to be a local diffeomorphism).

Theorem 7. *Let (M, g) be a Riemannian manifold. Let $\mathcal{E}$ be the domain of the exponential map, $p \in M$, $V \in \mathcal{E} \cap T_pM$, and $W \in T_pM$.*
Identifying $T_V(T_pM)$ with T_pM as usual, we compute the pushforward $(\exp_p)_*W$ at V as follows: set $\gamma(t) = \exp(tV)$, and let $J(t)$ be the Jacobi field along γ determined by the initial condition
$$J(0) = 0, \quad D_t J(0) = W.$$
Then for all t such that $tV \in \mathcal{E}$, we have
$$(\exp_p)_* \Big|_{tV} W = t^{-1} J_W(t),$$
and, in particular,
$$(\exp_p)_* \Big|_V W = J_W(1).$$
In other words, for all t such that $tV \in \mathcal{E}$, we have
$$J(t) = (d\exp_p)_{tV}(tD_t J(0)).$$

Proof. Identifying $T_V(T_pM)$ with T_pM as usual, we can compute the pushforward $(\exp_p)_*$ at V as follows:
$$(\exp_p)_*W = \frac{d}{ds}\Big|_{s=0} \exp_p(V + sW).$$
To compute this, we define a variation of γ through geodesics by
$$H_W(s, t) = \exp_p t(V + sW).$$
Then the variation field $J_W(t) = \partial_s H_W(0, t)$ is a Jacobi field along γ, and
$$J_W(t) = (\exp_p)_* \Big|_{tV} W. \qquad \square$$

Corollary 8. *The null space of $(\exp_p)_*\big|_V$ is isomorphic to the subspace of Jacobi fields along $\gamma(t) = \exp(tV)$ vanishing at p and $\exp(V)$.*

Conjugate Points

Definition. *If γ is a geodesic segment joining p, $q \in M$, q is said to be **conjugate to p along** γ if there is a Jacobi field along γ vanishing at p and q but not identically zero.*

- *The **order of multiplicity** of conjugacy is the dimension of the space of Jacobi fields vanishing at p and q.*

- From the existence and uniqueness theorem for Jacobi fields, there is an n-dimensional space of Jacobi fields that vanish at p;
  since tangential Jacobi fields vanish at most at one point, the order of conjugacy of two points p and q can be **at most** $n - 1$.

— This bound is sharp: if p and q are antipodal points on $\mathbb{S}_R^n$, there is a Jacobi field vanishing at p and q for each parallel normal vector field along γ; thus in that case p and q are conjugate to order exactly $n - 1$.

- The most important fact about conjugate points is that they are precisely the **images of singularities** of the exponential map.

Proposition 9. *Suppose $p \in M$, $V \in T_pM$, and $q = \exp_p V$. Then $\exp_p$ is a local diffeomorphism in a nbhd of V iff q is not conjugate to p along the geodesic $\gamma(t) = \exp_p(tV)$, $t \in [0, 1]$.*

Proof. By the inverse function theorem, $\exp_p$ is a local diffeomorphiem near V iff $(\exp_p)_*$ is an isomorphism at V, and by dimensional consideration, this occurs iff $(\exp_p)_*$ is injective at V.
Therefore, $(\exp)_*$ fails to be an isomorphism at V when there is a vector W such that $(\exp_p)_* W = 0$, which, by Corollary 8, occurs precisely when there is a Jacobi field J_W along γ with $J_W(0) = J_W(1) = 0$. $\quad\square$