Exterior Derivatives

- In this section we define a natural differential operator on smooth forms, called the exterior derivative.
  It is a generalization of the diffeential of a function.

Motivations:

- Recall that not all 1-forms are differentials of functions:
  Given a smooth 1-form ω, a necessary condition for the existence of a smooth function f such that $\omega = df$ is that ω be **closed**, which means that it satisfies

$$\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0 \tag{1}$$

  in every smooth coordinate system.

Proposition 1. *Let ω be a smooth covector field. If ω satisfies (1) in some smooth chart around every point, then it is closed.*

Proof. Let $(U, (x^i))$ be an arbitrary smooth chart. For each point $p \in U$, the hypothesis guarantees that there are some smooth coordinates $(\widetilde{x}^j)$ defined near p in which the analogue of (1) holds. We have

$$\begin{aligned}
\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} =& \frac{\partial}{\partial x^j}\left(\frac{\partial \widetilde{x}^k}{\partial x^i}\widetilde{\omega}^k\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial \widetilde{x}^k}{\partial x^j}\widetilde{\omega}^k\right) \\
=&\left(\frac{\partial^2 \widetilde{x}^k}{\partial x^i \partial x^j}\widetilde{\omega}^k + \frac{\partial \widetilde{x}^k}{\partial x^i}\frac{\partial \widetilde{\omega}^k}{\partial x^i}\right) - \left(\frac{\partial^2 \widetilde{x}^k}{\partial x^i \partial x^j}\widetilde{\omega}^k + \frac{\partial \widetilde{x}^k}{\partial x^j}\frac{\partial \widetilde{x}^k}{\partial x^i}\right) \\
=&\left(\frac{\partial^2 \widetilde{x}^k}{\partial x^i \partial x^j}\widetilde{\omega}^k + \frac{\partial \widetilde{x}^k}{\partial x^i}\frac{\partial \widetilde{x}^\ell}{\partial x^j}\frac{\partial \widetilde{\omega}^k}{\partial \widetilde{x}^\ell}\right) - \left(\frac{\partial^2 \widetilde{x}^k}{\partial x^i \partial x^j}\widetilde{\omega}^k - \frac{\partial \widetilde{x}^k}{\partial x^j}\frac{\partial \widetilde{x}^\ell}{\partial x^j}\frac{\partial \widetilde{x}^k}{\partial x^\ell}\right) \\
=&\left(\frac{\partial^2 \widetilde{x}^k}{\partial x^i \partial x^j} - \frac{\partial^2 \widetilde{x}^k}{\partial x^i \partial x^j}\right)\widetilde{\omega}^k + \frac{\partial \widetilde{x}^k}{\partial x^i}\frac{\partial \widetilde{x}^\ell}{\partial x^j}\left(\frac{\partial \widetilde{x}^k}{\partial x^\ell} - \frac{\partial \widetilde{\omega}^k}{\partial \widetilde{x}^\ell}\right) \\
=&0 + 0. \qquad \square
\end{aligned}$$

— By Proposition 1, **being a closed form** is a coordinate-independent property, and thus one might hope to find a more invariant way to express it.

— The key is that the expression in (1) is **antisymmetric** in the indices i and j, so it can be interpreted as the ij-component of an alternating tensor field, i.e. a 2-form.

— We will define a 2-form $d\omega$ by

$$d\omega = \sum_{i<j}\left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) dx^i \wedge dx^j,$$

so it follows that ω is closed iff $d\omega = 0$.

- This formula has a significant generalization to differential forms of all degrees.
- For any manifold, we will show that there is a differential operator

$$d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$$

  satisfying $d(d\omega) = 0$ for all ω.

— Thus it will follow that a **necessary** condition for a smooth k-form ω to be equal to $d\eta$ for some $(k-1)$-form η is that $d\omega = 0$.

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- The definition of d in coordinates is straightforward:

$$d\Big({\sum_J}' \omega_J dx^J\Big) = {\sum_J}' d\omega_J \wedge dx^J, \tag{2}$$

where $d\omega_J$ is just the differential of the function ω_J.

— In somewhat more detail, this is

$$d\Big({\sum_J}' \omega_J dx^{j_1} \wedge \cdots \wedge dx^{j_k}\Big) = {\sum_J}' \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j+k}. \tag{3}$$

— Observe that when ω is a 1-form, this becomes

$$\begin{aligned} d(\omega_j dx^j) &= \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i<j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j + \sum_{i>j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j. \end{aligned}$$

— For a smooth 0-form f (a real-valued function), (2) reduces to

$$df = \frac{\partial f}{\partial x^i} dx^i,$$

which is just the differential of f.

- Proving that this definition is **independent of the choice of coordinates** and thus **can be extended to smooth manifolds** takes a little work.

— This is the content of the next theorem.

Theorem 2 (The Exterior Derivative). *For every smooth manifold M, there are unique linear maps*

$$d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$$

defined for each $k \geq 0$ and satisfying the following three conditions:

(i) If f is a smooth, real-valued function (a 0-form), then df is the differential of f, defined as usual by

$$df(X) = Xf.$$

(ii) If $\omega \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^\ell(M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) $d \circ d = 0$.

This operator also satisfies the following properties:

(a) In every smooth coordinate chart, d is given by (2).

(b) d is local: If $\omega = \omega'$ on an open set $U \subset M$, then $d\omega = d\omega'$ on U.

(c) d commutes with restrictions: If $U \subset M$ is any open set, then

$$d(\omega\big|_U) = (d\omega)\big|_U. \tag{4}$$

Proof. **(I)** Begin with the **special case: Suppose M can be covered by a single smooth chart**.

⊙ Let $(x^1, \cdots, x^n)$ be global smooth coordinates on M, and define $d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ by (2).

— The map d thus defined is clearly linear and satisfies (i).

(I.1) Claim: It satisfies (ii) and (iii).

— Before doing so, we **claim**: d satisfies $d(f dx^I) = df \wedge dx^I$ for **any** multi-index I, not just increasing ones; indeed,

(1) if I has repeated indices, then clearly $d(f dx^I) = 0 = df \wedge dx^I$;

(2) if I has no repeated indices, then let σ be the permutation setting I to an increasing multi-index J, we have

$$d(f dx^I) = (\operatorname{sgn}\sigma) d(f dx^J) = (\operatorname{sgn}\sigma) df \wedge dx^J = df \wedge dx^I.$$

— **To prove (ii)**, by linearity it suffices to consider terms of the form $\omega = f dx^I$ and $\eta = g dx^J$. We compute

$$\begin{aligned} d(\omega \wedge \eta) =& d((f dx^I) \wedge (g dx^J)) \\ =& d(fg dx^I \wedge dx^J) \\ =& (g df + f dg) \wedge dx^I \wedge dx^J \\ =& (df \wedge dx^I) \wedge (g dx^J) + (-1)^k (f dx^I) \wedge (dg \wedge dx^J) \\ =& d\omega \wedge \eta + (-1)^k \omega \wedge d\eta, \end{aligned}$$

where the $(-1)^k$ comes from the fact that $dg \wedge dx^I = (-1)^k dx^I \wedge dg$ because dg is a 1-form and dx^I is a k-form.

— **Prove (iii) first for the special case of a 0-form**, i.e. a real-valued function. In this case,

$$\begin{aligned} d(df) =& d\left(\frac{\partial f}{\partial x^j} dx^j\right) = \frac{\partial^2 f}{\partial x_i \partial x_j} dx^i \wedge dx^j \\ =& \sum_{i<j} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right) dx^i \wedge dx^j = 0. \end{aligned}$$

For the general case, we use the $k = 0$ case together with (ii) to compute

$$\begin{aligned} d(d\omega) =& d\left(\sum_J{}' d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \right) \\ =& \sum_J{}' d(d\omega_J) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \\ &+ \sum_J{}' \sum_{i=1}^{k} (-1)^i d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge d(dx^{j_1}) \wedge \cdots \wedge dx^{j_k} \\ =& 0. \end{aligned}$$

This proves that there exists an operator d satisfying (i)-(iii) in the special case.

(I.2) Properties (a)-(c) are immediate consequences of the definition, once we note that if M is covered by a single smooth chart, then any subset of M has the same property.

(I.3) **To show that** d **is unique**, suppose $\tilde{d} : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ is another linear operator defined for each $k \geq 0$ and satisfying (i), (ii) and (iii).

— Let $\omega = \sum_J{}' \omega_J dx^J \in \mathcal{A}(M)$ be arbitrary.
Using linearity of $\tilde{d}$ together with (ii), we compute

$$\begin{aligned}\tilde{d}\omega =& \tilde{d}(\sum_J{}' \omega_J dx^{j_1} \wedge \cdots \wedge dx^{j_k}) \\ =& \sum_J{}' \tilde{d}\omega_J \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} + \sum_J{}' \omega_J \tilde{d}(dx^{j_1} \wedge \cdots \wedge dx^{j_k}).\end{aligned}$$

- Using (ii) again, the last term expands into a sum of terms, each of which contains a factor of the form $\tilde{d}(dx^{j_i})$, which is equal to $\tilde{d}(\tilde{d}x^{j_i})$ by (i) and hence is zero by (iii).
- On the other hand, since each component function ω_J is a smooth function, (i) implies that $\tilde{d}\omega_J = d\omega_J$.
- Thus $\tilde{d}\omega$ is equal to $d\omega$ defined by (2).
- This implies, in particular, that we obtain the same operator no matter which (global) smooth coordinates we use to define it.
- This completes the proof of the existence and uniqueness of d in the special case.

(II) Next, **let** M **be an arbitrarily smooth manifold**.

— On any smooth coordinate domain $U \subset M$, the argument above yields a unique linear operator from smooth k-forms to $(k+1)$-forms, which we denote by d_U, satisfying (i)-(iii).

— On any set $U \cap U'$ where two smooth charts overlap, the restrictions of $d_U\omega$ and $d_{U'}\omega$ to $U \cap U'$ satisfy

$$(d_U\omega)\Big|_{U\cap U'} = d_{U\cap U'}\omega = (d_{U'}\omega)\Big|_{U\cap U'}, \quad \text{by (4).}$$

— Therefore, we can unambiguously define $d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ by defining the value of $d\omega$ at $p \in M$ to be $(d\omega)_p = d_U(\omega|_U)_p$, where U is any smooth coordinate domain containing p.

— This operator satisfying (i), (ii), and (iii) because each d_U does.

- It also satisfies (a), (b), and (c) by definition.

(II.1) Finally, we **claim uniqueness in the general case**.

— Suppose we have some ther property

$$\widetilde{d} : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$$

defined for each k and satisfying (i)-(iii).

(II.1.1) Begin by **showing that** $\widetilde{d}$ **satisfies the locality property (b).**
For this, writing $\eta = \omega - \omega'$, it suffices to

claim: $\widetilde{d}\eta = 0$ **on an open set** U **if** $\eta\big|_U = 0$**.**

Indeed, for an arbitrary point $p \in U$, let $\varphi \in C^\infty(M)$ be a smooth bump function that is equal to 1 in a neighborhood of 1 and supported in U.
Then $\varphi\eta \equiv 0$, and hence

$$0 = \widetilde{d}(\varphi\eta)_p = \widetilde{d}\varphi_p \wedge \eta_p + \varphi(p)\widetilde{d}\eta_p = \widetilde{d}\eta_p,$$

because $\varphi \equiv 1$ in a neighborhood of p. Since p is an arbitrary point of U, this shows that $d\eta = 0$ on U.

(II.1.2) Let $U \subset M$ be an arbitrary smooth domain. For each k, define an operator

$$\widetilde{d}_U : \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)$$

as follows. For each $p \in U$,

(1) choose an extension of ω to a smooth global k-form $\widetilde{\omega} \in \mathcal{A}^k(M)$ that agrees with ω on a neighborhood of p, and
(2) set $(\widetilde{d}_U\omega)_p = (\widetilde{d}\widetilde{\omega})_p$.

Because $\widetilde{d}$ is local, this definition is independent of the extension $\widetilde{\omega}$ chosen.

— The fact that $\widetilde{d}$ satisfies (i)-(iii) implies immediately that $\widetilde{d}_U$ satsfies the same properties. This implies that

$$\widetilde{d}_U = d_U,$$

by the uniqueness property proved in **(I.3)** for smooth coordinate domains.

— In particular, if ω is the restriction to U of a global form $\widetilde{\omega}$ on M, then we can use the same extension $\widetilde{\omega}$ near each point, so

$$(d\widetilde{\omega})\Big|_U = d_U(\widetilde{\omega}\Big|_U) = \widetilde{d}_U(\widetilde{\omega}\Big|_U) = (\widetilde{d}\widetilde{\omega})\Big|_U.$$

This shows that $\widetilde{d}$ is equal to the operator d we defined above. $\square$

Definition. *The operator d whose existence and uniqueness are asserted in theorem is called* **exterior differentiation**, *and $d\omega$ is called the* **exterior derivative** *of ω.*

Definition. *If $A = \bigoplus_k A^k$ is a graded algebra, a linear map $T : A \to A$ is said to be* **of degree** *m if*

$$T(A^k) \subset A^{k+m} \quad \forall k.$$

- *It is said to be* **antiderivative** *if it satisfies $T(xy) = (Tx)y + (-1)^k x(Ty)$ whenever $x \in A^k$ and $y \in A^\ell$.*

Corollary. *The exterior differential extends to a antiderivative of $\mathcal{A}^*(M)$ of degree 1 whose square is zero.*

Definition. *(1) A smooth differential form $\omega \in \mathcal{A}^k(M)$ is said to be* **closed** *if $d\omega = 0$.*

(2) A smooth differential form $\omega \in \mathcal{A}^k(M)$ is said to be **exact** *if $\exists (k-1)$-form η on M such that $\omega = d\eta$.*

Corollary. *Every exact form is closed.*

Proof. This follows from $d \circ d = 0$. □

- One important feature of the exterior derivative is that **it behaves well w.r.t. pullbacks**, as the next lemma shows.

Lemma 3 (Naturality of the Exterior Derivative).. *If $G : M \to N$ is a smooth map, then the pullback map $G^* : \mathcal{A}^k(N) \to \mathcal{A}^k(M)$ commutes with d:*

$$G^*(d\omega) = d(G^*\omega), \quad \forall \omega \in \mathcal{A}^k(N). \tag{5}$$

Proof. Because d is local, it suffices, by linearity, to check (5) for a form of the type

$$f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

For such a form, the left-hand side of (5) is

$$\begin{aligned} G^* d(f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}) =& G^*(df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\ =& d(f \circ G) \wedge d(x^{i_1} \circ G) \wedge \cdots \wedge d(x^{i_k} \circ G), \end{aligned}$$

while the right-hand side is

$$\begin{aligned} dG^*(f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}) =& d((f \circ G) d(x^{i_1} \circ G) \wedge \cdots \wedge d(x^{i_k} \circ G)) \\ =& d(f \circ G) \wedge d(x^{i_1} \circ G) \wedge \cdots \wedge d(x^{i_k} \circ G). \end{aligned}$$ □