Laplacian And Harmonic Forms

- In this section, let (M, g) be an oriented Riemannian manifold, compact and without boundary.
  We note, however, that compactness is needed only when the formula (1) below is used.
- We defined the inner product $\langle \omega_p, \eta_p \rangle$ at each point for two k-forms on M.

— We now introduce the inner product in $\mathcal{A}^k(M)$ by integrating the function $\langle \omega_p, \eta_p \rangle$ over M; i.e.

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle \, v_M, \tag{1}$$

where v_M is the volume element of M. The following properties hold:
(i) (linearity) $(a\omega + b\omega', \eta) = a(\omega, \eta) + b(\omega', \eta)$.
(ii) (symmetry) $(\omega, \eta) = (\eta, \omega)$.
(iii) (positive-definite) $(\omega, \omega) \geq 0$; $(\omega, \omega) = 0$ iff $\omega = 0$.
Thus $(\cdot, \cdot)$ is an inner product on the vector space $\mathcal{A}^k(M)$.
In particular, the length $\|\omega\| = \sqrt{(\omega, \omega)}$ is defined.

- The inner product (1) can also be written in the form

$$(\omega, \eta) = \int_M \omega \wedge *\eta = \int_M \eta \wedge *\omega. \tag{2}$$

We have also

$$(*\omega, *\eta) = (\omega, \eta),$$

which means that **the Hodge operator** $* : \mathcal{A}^k(M) \to \mathcal{A}^{n-k}(M)$ **is isometric relative to the inner product above**.

— By convention, we define the inner product between differential forms of two different degrees to be 0,
so that the entire space $\mathcal{A}^*(M)$ is provided with an inner product.

- Next we study **how exterior differentiation** $d : \mathcal{A}^*(M) \to \mathcal{A}^*(M)$ **is transformed by the Hodge operator**.
- For this purpose, we consider linear operator

$$\delta^{k-1} = (-1)^k *^{-1} d* = (-1)^{n(k-1)+1} * d*$$

by requiring the following diagram be commutative

$$\begin{array}{ccc} \mathcal{A}^k(M) & \xrightarrow{\;*\;} & \mathcal{A}^{n-k}(M) \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ \mathcal{A}^{k-1}(M) & \xrightarrow[(-1)^k *]{} & \mathcal{A}^{n-k+1}(M). \end{array}$$

From the definition, we immediately see that

$$*\delta = (-1)^k d*, \quad \delta* = (-1)^{k^2+1} * d, \quad \delta \circ \delta = 0.$$

Proposition 1. *Relative to the inner product* $(\cdot,\cdot)$ *in* $\mathcal{A}^*(M)$, δ *is an* **adjoint operator** *of exterior differentiation d; o.e., we have*

$$(d\omega,\eta) = (\omega,\delta\eta). \tag{3}$$

Conversely, d is an adjoint operator of δ.

Proof. It suffices to prove (3) when ω and η are k- and $(k+1)$- forms, respectively. In this case, we have

$$\begin{aligned} d\omega \wedge *\eta =& d(\omega \wedge *\eta) - (-1)^k \omega \wedge d * \eta \\ =& d(\omega \wedge *\eta) + \omega \wedge *\delta\eta. \end{aligned}$$

Integrating each side over M, we obtain from (2)

$$(d\omega,\eta) = \int_M d(\omega \wedge *\eta) + (\omega,\delta\eta) = (\omega,\delta\eta). \qquad \square$$

Definition. *For a Riemannian manifold M, the operator defined by*

$$\Delta = d\delta + \delta d : \mathcal{A}^k(M) \to \mathcal{A}^k(M)$$

is called the **Laplacian** *or* **Laplace-Beltrami operator**.

- *A form* $\omega \in \mathcal{A}^*(M)$ *such that* $\Delta\omega = 0$ *is called a* **harmonic form**. *In particular, a function such that* $\Delta f = 0$ *is called a* **harmonic function**.

Example. To compute the Laplacian on $\mathbb{R}^n$, it suffices to compute $\Delta\omega$ for a k-form written

$$\omega = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $x_1, \cdots, x_n$ are the ordinary coordinates in $\mathbb{R}^n$.

– First, choose $j_1, \cdots, j_{n-k}$ such that

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}} = dx_1 \wedge \cdots \wedge dx_n.$$

Then we obtain, by the definition of $*$,

$$*\omega = f\, dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}}$$

$$d * \omega = \sum_{s=1}^{k} \frac{\partial f}{\partial x_{i_s}}\, dx_{i_s} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}},$$

$$\delta\omega = (-1)^{n(k+1)+1} * d * \omega = \sum_{s=1}^{k} \frac{\partial f}{\partial x_{i_s}} dx_{i_s} \wedge dx_{i_1} \wedge \cdots \widehat{dx_{i_s}} \wedge \cdots \wedge dx_{i_k}.$$

Therefor we obtain

$$d\delta\omega = -\sum_{s=1}^{k} \frac{\partial^f}{\partial x_{i_s}^2} dx_{i_s} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} + (-1)^s \frac{\partial^2 f}{\partial x_{i_s} \partial x_{j_t}} dx_{j_t} \wedge dx_{i_s} \wedge dx_{i_1} \wedge \cdots \widehat{dx_{i_s}} \wedge \cdots \wedge dx_{i_k}. \tag{4}$$

— On the other hand, we have

$$d\omega = \sum_{s=1}^{n-k} \frac{\partial f}{\partial x_{j_s}} dx_{j_s} \wedge dx_{j_s} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

and thus

$$*d\omega = \sum_{s=1}^{n-k} (-1)^{k+s-1} \frac{\partial f}{\partial x_{j_s}} dx_{j_1} \wedge \widehat{dx_{j_s}} \wedge \cdots \wedge dx_{j_{n-k}}.$$

Further, we have

$$d * d\omega = \sum_{s=1}^{n-k} (-1)^k \frac{\partial^2 f}{\partial x_{x_{j_s}}^2} dx_{j_1} \cdots \wedge dx_{j_{n-k}} + \sum_{s=1}^{n-k} \sum_{t=1}^{k} (-1)^{k+s-1} \frac{\partial^2 f}{\partial x_{j_s} \partial x_{i_s}} dx_{i_s} \wedge dx_{j_1} \wedge \widehat{dx_{j_s}} \wedge \cdots \wedge dx_{j_{n-k}},$$

and hence

$$\delta d\omega = -\sum_{s=1}^{n-k} \frac{\partial^2 f}{\partial x_{x_{j_s}}^2} dx_{i_1} \cdots \wedge dx_{i_k} + \sum_{s=1}^{n-k} \sum_{t=1}^{k} (-1)^{t+1} \frac{\partial^2 f}{\partial x_{j_s} \partial x_{i_s}} dx_{j_s} \wedge dx_{i_1} \wedge \widehat{dx_{i_t}} \wedge \cdots \wedge dx_{i_k}. \tag{5}$$

— Adding (4) and (5), we arrive at

$$\Delta\omega = -\sum_{s=1}^{n-k} \frac{\partial^2 f}{\partial x_{x_{j_s}}^2} dx_{i_1} \cdots \wedge dx_{i_k}.$$

Thus the Laplacian is an extension of the classical Laplace operator to the case of a general Riemannian manifold.

Proposition 2. *The Laplacian Δ has the following properties:*
(i) $*\Delta = \Delta *$. If ω is a harmonic form, so is $*\omega$.
(ii) Δ is self-adjoint, that is,

$$(\Delta\omega, \eta) = (\omega, \Delta\eta), \quad \forall \omega, \eta \in \mathcal{A}^*(M).$$

(iii) $\Delta\omega = 0$ iff $d\omega = 0$ and $\delta\omega = 0$.

Proof. (i) is simple.
(ii) follows from the fact that d and δ are adjoint to each other.
(iii) ($\Leftarrow$) If $d\omega = \delta\omega = 0$, then clearly $\Delta\omega = 0$.
($\Rightarrow$) To show the converse, we need the assumption that M **is compact**.
In this case, the equality

$$0 = (\Delta\omega, \omega) = ((d\delta + \delta d)\omega, \omega) = (\delta\omega, \delta\omega) + (d\omega, d\omega) = 0$$

shows that $\Delta\omega = 0$ implies $\|\delta\omega\| = \|d\omega\| = 0$, i.e., $d\omega = \delta\omega = 0$. □

Corollary 3. *If* (M, g) *is a connected, oriented* **compact** *Riemannian* n*-manifold, then (i) a harmonic function on* M *is a constant function, and*
(ii) a harmonic n*-form is a constant multiple of the volume element* dv_g*.*

Proof. (i) If a function f satisfies $\Delta f = 0$, then Proposition 2(iii) implies $df = 0$. Hence f is a constant function if M is connected.
(ii) Any n-form on M must be a function times the volume element:

$$\omega = f\, dv_g.$$

If $\Delta\omega = 0$, then

$$*\omega = *(f\, dv_g) = f$$

is a harmonic function, hence f =constant c; i.e. $\omega = c\, dv_g$. □

- Let M be an oriented **compact** Riemannian manifold.

— If r is the number of connected components, then
both $H^0_{dR}(M)$ and $H^n_{dR}(M)$ are isomorphic to the derect sum of r copies of $\mathbb{R}$.

— From Corollary 3, it follows that for $k = 0$ and for $k = n$,
every element of H^k_{dR} is represented by a uniquely determined harmonic form.

– This fact remains valid for every k, as we will see in the Hodge theorem.