Bochner-Weitzenböck Formula for Functions

(I) If $f : M \to \mathbb{R}$ is smooth, denote $\nabla f = \text{grad}\, f$, we have

$$g(\nabla f, X) = df(X).$$

The $(1,1)$**-Hessian** $\nabla^2 f$ is defined as the $(1,1)$-tensor $D(\nabla f)$.
This tensor is self-adjoint, or symmetric, since

$$\begin{aligned} g(\nabla^2 f(X), Y) =& \langle D_X \nabla f, Y\rangle \\ =& D_X\langle \nabla f, Y\rangle - \langle \nabla X, D_X Y\rangle \\ =& D_X df(Y) - df(D_X Y) \\ =& X(Y(f)) - df(D_X Y) \\ =& X(Y(f)) - df(D_Y X) - df([X,Y]) \\ =& X(Y(f)) - X(Y(f)) + Y(X(f)) - df(D_Y X) \\ =& g(\nabla^2 f(Y), X). \end{aligned}$$

Thus, $\nabla^2 f$ can also be interpreted as the symmetric $(0,2)$-tensor

$$\nabla^2 f(X,Y) = g(\nabla^2 f(X), Y).$$

- The Laplacian

$$\Delta f = -\text{tr}\,(\nabla^2 f).$$

⊙ The divergence of s vector field X on (M, g) is defined as

$$\text{div}\, X = -\text{tr}\,(DX) = -\sum_{i=1}^{n} g(D_{e_i} e_i, e_i),$$

if $\{e_i\}$ is an orthonormal basis. Thus, also

$$\Delta f = \text{tr}\,(\nabla^2 f) = \text{tr}(D(\nabla f)) = \text{div}\,(\nabla f).$$

(II) If S is a $(1,1)$ tensor we want to have

$$D_X(S(Y)) = (D_X S)(Y) + S(D_X Y).$$

Thus it is reasonable to define DS as

$$\begin{aligned} DS(X,Y) =& (D_X S)(Y) \\ X(S(Y)) - S(D_X Y). \end{aligned}$$

More generally, we define

$$\begin{aligned} DS(X, Y_1, \cdots, Y_r) =& (D_X S)(Y_1, \cdots, Y_r) \\ =& D_X(S(Y_1, \cdots, Y_r)) - \sum_{i=1}^{r} S(Y_1, \cdots, D_X Y_i, \cdots, Y_r), \end{aligned}$$

where D_X is interpreted as the directional derivative when applied to a function.

- For a $(\cdot, r)$-tensor field S we can now also define the **second covariant derivative** $D^2 S$ as the $(\cdot, r+2)$-tensor field

$$\begin{aligned} (D^2_{X_1,X_2} S)(X, Y_1, \cdots, Y_r) =& (D_{X_1}(DS))(X_2, Y_1, \cdots, Y_r) \\ =& (D_{X_1} D_{X_2} S)(Y_1, \cdots, Y_r) - (D_{D_{X_1} X_2} S)(Y_1, \cdots, Y_r), \end{aligned}$$

With this we obtain for $f \in C^\infty(M)$,

$$\begin{aligned} D^2_{X,Y} f =& D_X D_Y f - D_{D_X Y} f \\ =& D_X g(Y, \nabla f) - g(D_X Y \nabla f) \\ =& g(Y, D_X \nabla f) \\ =& g(\nabla^2 f(X), Y). \end{aligned}$$

From this new formula for the Hessian, we see that the Laplacian can be written as

$$\Delta f = -\sum_{i=1}^{n} D^2_{e_i,e_i} f.$$

For a smooth function f, we define its gradient, Hessian, and the Laplacian by

$$\begin{aligned}\langle \nabla f, X\rangle =& X(f) = df(X),\\ \operatorname{Hess} f(X,Y) =& \langle D_X(\nabla f), Y\rangle,\\ \Delta f =& -\operatorname{tr}(\operatorname{Hess}\ f).\end{aligned}$$

For a bilinear form A, we write $|A|^2 = \operatorname{tr}(AA^t)$.

Theorem (the Bochner-Weitzenböck formula). *Let (M,g) be a complete Riemannian manifold. Then, $\forall f \in C^\infty(M)$, we have*

$$-\frac{1}{2}\Delta|\nabla f|^2 = |\operatorname{Hess} f|^2 - \langle \nabla f, \nabla(\Delta f)\rangle + \operatorname{Ric}(\nabla f, \nabla f).$$

Proof. Fix a point $p \in M$. Let $\{X_i\}_1^n$ be a local orthonormal frame field such that

$$\langle X_i, X_j\rangle = \delta_{ij}, \quad D_{X_i}X_j(p) = 0.$$

Computation at p gives

$$\begin{aligned}-\frac{1}{2}\Delta|\nabla f|^2 &= \frac{1}{2}\sum_i X_iX_i\langle \nabla f, \nabla f\rangle\\ &= \sum_i X_i\langle D_{X_i}\nabla f, \nabla f\rangle\\ &= \sum_i X_i \operatorname{Hess} f(X_i, \nabla f)\\ &= \sum_i X_i \operatorname{Hess} f(\nabla f, X_i) \quad (\because \text{ Hessian is symmetric}),\\ &= \sum_i X_i\langle D_{\nabla f}(\nabla f), X_i\rangle\\ &= \sum_i \langle D_{X_i}D_{\nabla f}(\nabla f), X_i\rangle + \langle D_{\nabla f}(\nabla f), D_{X_i}X_i\rangle\\ &= \sum_i \langle D_{X_i}D_{\nabla f}(\nabla f), X_i\rangle \quad (\because \text{ the other term vanishes at } p)\\ &= \sum_i \langle R(\nabla f, X_i)\nabla f, X_i\rangle + \sum_i \langle D_{\nabla f}D_{X_i}\nabla f, X_i\rangle + \sum_i \langle D_{[X_i,\nabla f]}\nabla f, X_i\rangle.\end{aligned}$$

The first term is by definition $\operatorname{Ric}(\nabla f, \nabla f)$.
The second term is

$$\begin{aligned}&\sum_i \langle D_{\nabla f}D_{X_i}\nabla f, X_i\rangle\\ &= \sum_i (\nabla f)\langle D_{X_i}\nabla f, X_i\rangle - \sum_i \langle D_{X_i}\nabla f, D_{\nabla f}X_i\rangle\\ &= (\nabla f)\sum_i \langle D_{X_i}\nabla f, X_i\rangle - 0 \ \ (\text{at } p)\\ &= -(\nabla f)\Delta f\\ &= -\langle \nabla f, \nabla(\Delta f)\rangle.\end{aligned}$$

The third term is

$$
\begin{aligned}
&\sum_i \langle D_{[X_i,\nabla f]}\nabla f, X_i\rangle \\
&= \sum_i \operatorname{Hess}(f)([X_i,\nabla f], X_i) \\
&= \sum_i \operatorname{Hess}(f)(D_{X_i}\nabla f - D_{\nabla f}X_i, X_i) \\
&= \sum_i \operatorname{Hess}(f)(D_{X_i}\nabla f, X_i) - \sum_i \operatorname{Hess}(f)(D_{\nabla f}X_i, X_i) \\
&= \sum_i \operatorname{Hess}(f)(D_{X_i}\nabla f, X_i) - 0 \ \text{ (at } p) \\
&= \sum_i \operatorname{Hess}(f)(D_{X_i}\nabla f, X_i) \\
&= \langle D_{X_i}\nabla f, D_{X_i}\nabla f\rangle \\
&= |\operatorname{Hess}(f)|^2. \qquad \square
\end{aligned}
$$

Proposition 4.15 (the Bochner-Weitzenböck formula). *Let (M, g) be a complete Riemannian manifold. Then, $\forall f \in C^\infty(M)$, we have*

$$\langle \Delta(df), df\rangle = \frac{1}{2}\Delta|df|^2 + |Ddf|^2 + \operatorname{Ric}(\nabla f, \nabla f).$$

- The power of this formula is that we have the freedom to choose the function f. Most of the results of comparison geometry are obtained by choosing f to be the distance function, the eigenfunction, and the displacement function, among others.
- We will consider the distance function.