Stokes' Theorem on Chains.

- Let M be a C^∞ manifold and $S_*^\infty(M) = \{S_k^\infty(M), \partial\}$ the C^∞ singular chain complex of M.

⊙ Recall that a C^∞ singular k-simplex $\sigma : \Delta^k \to M$ of M is a C^∞ map from (an open neighborhood of) the standard simplex Δ^k in $\mathbb{R}^k$ to M.
Therefore, for a k-form $\omega \in \mathcal{A}^k(M)$ on M, the pullback $\sigma^*\omega$ of ω by σ is defined.

— Now we define the integral of ω on σ by

$$\int_\sigma \omega = \int_{\Delta^k} \sigma^*\omega.$$

— For a general chain $c \in S_k^\infty(M)$, we extend this definition linearly.
That is, if $c = \sum_i a_i\sigma_i$, we set

$$\int_c \omega = \sum_i a_i \int_{\sigma_i} \omega.$$

Stokes' Theorem on Chains. *For a C^∞ singular k-chain $c \in S_k^\infty(M)$ of a C^∞ manifold M and a $(k-1)$-form ω on M, we have*

$$\int_c d\omega = \int_{\partial c} \omega.$$

Proof. By the linearity of the integral, it is enough to prove the case where c is a single singular k-simplex σ. Furthermore, since $\sigma^*\omega$ is a $(k-1)$-form on Δ^k, we can write

$$\sigma^*\omega = \sum_{i=1}^{k} a_i(x) dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx^k.$$

Again by linearity of the integral, it is enough to prove the case of

$$\sigma^*\omega = a(x) dx^1 \wedge \cdots dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^k.$$

Then we have

$$\sigma^* d\omega = (-1)^{j-1} \frac{\partial a}{\partial x^j} dx^1 \wedge \cdots dx^k,$$

For $i = 0, 1, \cdots, k$, define the continuous maps $\epsilon_i : \Delta^{k-1} \to \Delta^k$ by

$$\epsilon_0(x_1, \cdots, x_{k-1}) = \left(1 - \sum_{i=1}^{k-1} x_i, x_1, \cdots, x_{k-1}\right), \tag{1}$$

$$\epsilon_i(x_1, \cdots, x_{k-1}) = (x_1, \cdots, x_{i-1}, 0, x_i, \cdots, x_{k-1}) \quad (i = 1, \cdots, k); \tag{2}$$

using these, the boundary operator

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i \sigma \circ \epsilon_i.$$

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Thus the formula to be proved is

$$(3)\quad (-1)^{j-1}\int_{\Delta^k}\partial_j a dx^1\cdots dx^k=\sum_{i=0}^{k}(-1)^i\int_{\Delta^{k-1}}\epsilon_i^*(a(x)dx^1\wedge\cdots\wedge\widehat{dx_j}\wedge dx^k),$$

where $(\Delta')^{k-1}$ is the standard $(k-1)$-simplex in $(k-1)$-dimensional space obtained by omitting the x_j-direction from $\mathbb{R}^k$.

(i) The integral on the left hand side of (3)

$$\begin{aligned}\int_{\Delta^k}\partial_j a\,dx^1\cdots dx^k&=\int_{(\Delta')^{k-1}}\Big(\int_0^{1-\sum_{i\neq j}x_i}\partial_j a\,dx^j\Big)dx^1\cdots\widehat{dx^j}\cdots dx^k\\ (4)\qquad&=\int_{(\Delta')^{k-1}}\big(a(x^1,\cdots,x^{j-1},1-\sum_{i\neq j}x_i,x^j,\cdots,x^{k-1})\\ &\quad-a(x^1,\cdots,x^{j-1},0,x^j,\cdots,x^{k-1})\big)dx^1\cdots\widehat{dx^j}\cdots dx^k,\end{aligned}$$

where $(\Delta')^{k-1}$ is the standard $(k-1)$-simplex in $(k-1)$-dimensional space obtained by omitting the x_j-direction from $\mathbb{R}^k$.

— Identifying $(\Delta')^{k-1}$ and Δ^{k-1} via the diffeomorphism $\psi:(\Delta')^{k-1}\to\Delta^{k-1}$,

$$\psi(x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_{k-1},x_k)=(x_1,\cdots,x_{j-1},x_k,x_{j+1,},x_{k-1}),$$

whose determinant of the Jacobian matrix is $(-1)^{j-1}$, the last integral

$$\begin{aligned}&\int_{(\Delta')^{k-1}}\big(a(x^1,\cdots,x^{j-1},1-\sum_{i\neq j}x_i,x^j,\cdots,x^{k-1})\\ (5)\qquad&\quad-a(x^1,\cdots,x^{j-1},0,x^j,\cdots,x^{k-1})\big)dx^1\cdots\widehat{dx^j}\cdots dx^k\\ &=\int_{\Delta^{k-1}}\big(a(x^1,\cdots,x^{j-1},1-\sum_{i\neq j}x_i,x^j,\cdots,x^{k-1})\\ &\quad-a(x^1,\cdots,x^{j-1},0,x^j,\cdots,x^{k-1})\big)dx^1\cdots dx^{k-1}.\end{aligned}$$

(ii) By (1) and (2), we see that $\epsilon^*dx_i\neq 0$ only if $i=0$ or $i=j$, and hence the right-hand side od (3) is

$$\begin{aligned}&\sum_{i=0}^{k}(-1)^i\int_{\Delta^{k-1}}\epsilon_i^*(a(x)dx^1\wedge\cdots\wedge\widehat{dx_j}\wedge dx^k)\\ (6)\qquad&=(-1)^{j-1}\int_{\Delta^{k-1}}a(1-\sum_{i=1}^{k-1}x_i,x_1,\cdots,x_{k-1})dx^1\cdots dx^{k-1}\\ &\quad+(-1)^{j-1}\int_{\Delta^{k-1}}a(x^1,\cdots,x^{j-1},0,x^j,\cdots,x^{k-1})dx^1\cdots dx^{k-1}\end{aligned}$$

— Define a diffeomorphism $\varphi:\Delta^{k-1}\to\varphi(\Delta^{k-1})$ by

$$\varphi(x_1,\cdots,x_{k-1})=(x^1,\cdots,x^{j-1},1-\sum_{i=1}^{k-1}x_i,x^j,\cdots,x^{k-1}),$$

which transforms Δ^{k-1} onto a face of Δ^k and whose determinant of the Jacobian metric is $(-1)^{j-1}$. Therefore,

$$
\begin{aligned}
&(-1)^{j-1}\int_{\Delta^{k-1}} a(1-\sum_{i=1}^{k-1} x_i, x_1, \cdots, x_{k-1})\, dx^1\cdots dx^{k-1} \\
&=(-1)^{j-1}\int_{\Delta^{k-1}} a(x_1, \cdots, x_{j-1}, 1-\sum_{i=1}^{k-1} x_i, x_j, \cdots, x_{k-1})\, dx^1\cdots dx^{k-1}
\end{aligned}
\tag{7}
$$

From (4), (5), (6) and (7) we obtain (3). $\square$