Riemannian measure

- Start with the formula for change of variables of integral calculus:
  Let D, Ω be domains in $\mathbb{R}^n$, $n \geq 1$, and let

$$\varphi : D \to \Omega$$

  be a C^1 diffeomorphism, and let $J_\varphi(x)$ denote the Jacobian matrix associated to φ at x. Then, for any L^1 function f on Ω, we have

$$\int_D (f \circ \varphi)|\det J_\varphi| dV = \int_\Omega f \, dV. \tag{7}$$

- Now let M be a Riemannian manifold, and let $x : U \to \mathbb{R}^n$ be a chart on M. Then, for each $p \in U$, we let $G^x(p)$ denote the matrix given by

$$G^x(p) = (g^x_{ij}(p)), \quad g^x_{ij}(p) = \left\langle \frac{\partial}{\partial x^i}\bigg|_p, \frac{\partial}{\partial x^j}\bigg|_p \right\rangle,$$

  and we set

$$g^x = \det G^x > 0.$$

Question: What if we are given a different chart $y : U \to \mathbb{R}^n$ on the same set U in M?

Then we relate the formulae as follows: Set J to be the Jacobian matrix

$$J_{rj} = \frac{\partial (y \circ x^{-1})^r}{\partial x^j};$$

then we have

$$\frac{\partial}{\partial x^j} = \sum_r \frac{\partial}{\partial y^r} J_{rj}$$

which implies

$$G^x = J^T G^y J,$$

where J^T denotes the transpose of J, which implies

$$\sqrt{g^x} = \sqrt{g^y} |\det J|.$$

Thus we have the local densities

$$\sqrt{g^x} dx^1 \cdots dx^n = \sqrt{g^y} dy^1 \cdots dy^n, \tag{8}$$

by which we mean that the integral

$$I(f;U) = \int_{x(U)} (f\sqrt{g^x}) \circ x^{-1} dx^1 \cdots dx^n$$

depends only on f and U—not on the particular choice of chart x.

Typeset by $\mathcal{AMS}$-TEX

- We now turn the local Riemannian measure to a **global** Riemannian measure on all of M.

(i) First, pick an atlas
$$\{x_\alpha : U_\alpha \to \mathbb{R}^n : \alpha \in A\},$$
and subordinate partition of unity $\{\phi_\alpha : \alpha \in A\}$.

(ii) Then define the global Riemannian metric dV by
$$dV := \sum_{\alpha \in A} \phi_\alpha \sqrt{g^{x_\alpha}} dx_\alpha^1 \cdots dx_\alpha^n,$$
or, eqivalently,
$$\int_M f\, dV = \sum_{\alpha \in A} I(\phi_\alpha; f; U_\alpha).$$

— One easily check that the measure is **well-defined**; that is, it is independent of both the particular choices of atlas and subordinate partition of unity.

— One easily checks that a function f is measurable w.r.t. dV iff $f \circ x^{-1}$ is measurable on $x(U)$ for any chart $x : U \to \mathbb{R}^n$.

In what follws, we work with this measure.

Definition. *For any measurable B in M, we let $V(B)$ denote the measure of B and refer to $V(B)$ as the* **volume** *of B.*
If Γ is an $(n-1)$-dimensional submfd of M, then we usually denote its Riemannian measure by dA;
and for any measurable Λ in Γ, we denote its measure by $A(\Lambda)$, and refer to $A(\Lambda)$ as the area of Λ.

The Effective Calculation of Integrals

- If the manifold M is diffeomorphic to $\mathbb{R}^n$, then one has, possibly, a convenient way to literally calculate an integral, by referring the calculation to one coordinate system.
  However, as soon as one cannot cover the manifold with one "naturally" chosen chart, one would then be forced to literally pick an atlas and subordinate partition of unity.
  This would not go well at all.

- The simplest overarching approach is to use the geometry of the Riemannian manifold to indicate **a judicious choice of a set of measure** 0 **to delete**, which will thereby leave an open set that is the domain of a chart on M.

— The quickest example that comes to mind is the stereographic projection of the sphere $\mathbb{S}^n$ to $\mathbb{R}^n$, in which the domain of the chart covers all of $\mathbb{S}^n$ minus the pole of the projection.
So any integral on the sphere may be referred to this chart.

- Before proceeding, we note that (8) implies that **the notion of a set of measure** 0 depends only on the differentiable structure of the mfd.

— It makes no difference whether we are referring to a local measure on M induced by Lebesgue measure on the image of a chart on M, or whether we are referring to Riemannian measure.

- Continung, we work, in our setting, with **polar coordinates** as follows.
  For convenience, we assume that M is complete.
  For any $p \in M$, introduce normal coordinates about p, which describe (locally) a differentiable map of $(0, \infty) \times \partial B_1(0_p)$ into $M \setminus \{p\}$, given by
$$(t, \xi) \mapsto \exp t\xi.$$
  (i) This map may fail to be the inverse map of a chart on $M \setminus \{p\}$ since the map may fail to be a diffemorphism.
  (ii) Also, since $\exp(\partial B_1(0_p))$ is not diffeomorphic to a subset of $\mathbb{R}^{n-1}$, one cannot use ξ, literally, as an $(n-1)$-dimensional coordinate.
  - **The second difficulty** is simply addressed by picking a chart on $\exp(\partial B_1(0_p)$.
    It need never be explicit, since the final formulation never require it.
  - **The first difficulty** must be dealt with by restricting the geodesic spherical coordinates to $\mathbf{D}_p \setminus \{p\}$.
- Thus, a chart on $M \setminus \mathrm{Cut}(p) = D(p)$ is given by
$$\left(\exp\Big|_{D_p \setminus \{p\}}\right)^{-1} : D_p \setminus \{p\} \to \mathbf{D}_p \setminus \{p\};$$
  and the Riemannian measure is given on D_p by
$$dV(\exp \xi) = \sqrt{g}(t; \xi) dt\, d\mu_p(\xi),$$
  for some function $\sqrt{g}$ on $\mathbf{D}_p$, where $d\mu_p(\xi)$ denote the Riemannian measure on $\partial B_1(0_p)$ induced by the Eucliden Lebesgue measure on T_pM.

(i) The set $\{p\}$ has measure 0; so we never have to explicitly include it in, or exclude it from, our discussion of integrals.
(ii) More significantly, $C(p)$ has measure 0. Indeed, the function $c(\xi)$ is continuous on all of $\mathbf{S}M = \{\xi \in TM : |\xi| = 1\}$, so its restriction to $\partial B_1(0_p)$ is continuous.

- Thus, the tangential cut locus of p is the image of the continuous map
$$\xi \mapsto c(\xi)\xi$$
  from $\partial B_1(0_p)$ to T_pM, and therefore has the Lebesgue measure 0.
  The image of the tangential cut locus of p under the differentiable exponential map is $C(p)$, the cut locus of p in M. Therefore

Proposition 1. *For any $p \in M$, the cut locus $C(p)$ of p is a set of measure 0.*

Thus, for any $p \in M$, and integrable function f on M, we have
$$\begin{aligned}\int_M f dV &= \int_{\mathbf{D}_p} f(\exp t\xi)\sqrt{g}(t;\xi) dt\, d\mu_p(\xi) \\ &= \int_{\partial B_1(0_p)} d\mu_p(\xi) \int_0^{c(\xi)} f(\exp t\xi)\sqrt{g}(t;\xi) dt \\ &= \int_0^{\infty} dt \int_{t^{-1}\partial B_t(0_p) \cap \mathbf{D}_p} f(\exp t\xi)\sqrt{g}(t;\xi) d\mu_p(\xi)\end{aligned}$$
where $t^{-1}\partial B_t(0_p) \cap \mathbf{D}_p$ is the subset of $\partial B_1(0_p)$ obtained by dividing each of the elements of $\partial B_t(0_p) \cap \mathbf{D}_p$ by t.

Theorem 2. *We have*

$$\sqrt{g}(t;\xi) = \det \mathcal{A}(t;\xi),$$

where $\mathcal{A}(t;\xi)$ is the solution of the system of ODEs on $\xi^{\perp}$:

$$\mathcal{A}'' + \mathcal{R}(t)\mathcal{A} = 0,$$

satisfying the initial conditions $\mathcal{A}(0;\xi) = 0$, $\mathcal{A}'(0;\xi) = I$.

Proof. Let ψ be a chart on $\partial B_1(0_p)$, $\xi = \psi^{-1}$, and let x be a chart on $D_p \setminus \{p\}$ given by

$$x = \left(\psi \circ \left[\frac{(\exp\big|_{D_p})^{-1}}{|(\exp\big|_{D_p})^{-1}|} \right], |(\exp\big|_{D_p})^{-1}| \right).$$

Then, what was called $\partial_t H$ is here equal to $\partial/\partial x^n$, and what was referred to as $\partial_\alpha H$ is here equal to $\partial/\partial x^\alpha$, $\alpha = 1, \cdots, n-1$.

— Let G be the matrix of the Riemannian metric on M associated to the chart x, and let $\widehat{G}$ be the matrix of the Riemannian metric on $\partial B_1(0_p)$ associated to the chart u.

— Then, equation (6) translates to our language here as

$$g_{\alpha\beta} = \sum_{\gamma,\delta} \mathcal{A}^*_{\alpha\gamma} \widehat{g}_{\gamma\delta} \mathcal{A}_{\delta\beta}, \quad \alpha, \beta, \gamma, \delta = 1, \cdots, n-1,$$

and (4) and (5) translate to

$$g_{nn} = 1, \quad g_{\alpha n} = g_{n\alpha} = 0, \quad \alpha = 1, \cdots, n-1.$$

We conclude that

$$\sqrt{g} = \sqrt{\widehat{g}} \det \mathcal{A},$$

which implies the claim. □

Notation. Given $x \in M$, we let $V(x;r)$ denote the volume of $B_r(x) = B(x;r)$; that is

$$V(x;r) = \int_{B_r(x)} dV.$$

Notation. For each $x \in M$, $r > 0$, define $\mathbf{D}_x(r)$ to be the subset of $\partial B_1(0_x)$ consisting of those elements ξ for which $r\xi \in \mathbf{D}_x$, i.e.

$$r\mathbf{D}_x(r) = \partial B_r(0_x) \cap \mathbf{D}_x.$$

— We have

$$\begin{aligned} V(x;r) &= \int \int_{\mathbf{D}_x \cap B_r(x)} \det \mathcal{A}(t;xi) dt d\mu_x(\xi) \\ &= \int_0^r dt \int_{\mathbf{D}_x(t)} \det \mathcal{A}(t;\xi) d\mu_x(\xi). \end{aligned}$$