Harmonic 1-Forms (The Bochner Technique)

- Suppose that ω is a harmonic 1-form on (M, g). We shall consider

$$f = \frac{1}{2} g(\omega, \omega).$$

  One of the technical problems is that we do not have a really good feeling for what $g(\omega, \omega)$ is.

– If X is the vector field dual to ω, i.e. $\omega(v) = g(X, v)$ for all $v \in \Gamma(TM)$, then,

$$f = \frac{1}{2} g(\omega, \omega) = \frac{1}{2} g(X, X) = \frac{1}{2} \omega(X).$$

Proposition 1. *If X is a vector field on (M, g) and $\omega(v) = g(X, v)$ is the dual 1-form, then*

$$\mathit{div}\, X = -\delta\omega.$$

Proposition 2. *If X is a vector field on (M, g) and $\omega(v) = g(X, v)$ is the dual 1-form, then $v \mapsto D_v X$ is symmetric iff $d\omega = 0$.*

Proof. Recall that

$$d\omega(V, W) = D_V \omega(W) - D_W \omega(V) - \omega([V, W]).$$

Using that $\omega(Z) = g(X, Z)$, we then obtain

$$\begin{aligned} d\omega(V, W) =& D_V g(X, W) - D_W g(X, V) - g() - g(X, D_V W) + g(X, D_W V) \\ =& [g(D_V X, W) + g(X, D_V W)] - [g(D_W X, V) + g(X, D_W V)] \\ & - g(X, D_V W) + g(X, D_W V) \\ =& g(D_V X, W) - g(D_W X, V). \qquad \square \end{aligned}$$

Corollary. *If ω is harmonic and X is the dual vector field, we have that $\mathit{div}\, X = 0$ and DX is a symmetric $(1, 1)$-tensor.*

Proposition 3. *Let X be a vector field so that DX is stmmetric. If we define a symmetric $(1, 1)$-tensor by $S(v) = D_v X$, then*
(a) $\frac{1}{2} \nabla |X|^2 = D_X X$,
(b) $\frac{1}{2} D_V \nabla |X|^2 = R(X, V)X + (D_X S)(V) + S^2(V)$.
(c) $\frac{1}{2} \Delta |X|^2 = \Delta f = \mathrm{Ric}(X, X) + g(X, \nabla \mathrm{div}\, X) + |DX|^2$.

Proof. (a) Let $f = \frac{1}{2}|X|^2$. Observe that, for all $V \in \Gamma(TM)$,

$$\begin{aligned} g(\nabla f, V) = df(V) =& Vf = D_V \frac{1}{2} g(X, X) \\ =& g(D_V X, X) \\ =& g(D_X X, V), \quad \text{since } DX \text{ is symmetric.} \end{aligned}$$

(b) Note that

$$\begin{aligned}\frac{1}{2}D_V\nabla|X|^2 =& D_V D_X X, \quad \text{by (a)}\\ =& R(X,V)X + D_X D_V X + D_{[V,X]}X\\ =& R(X,V)X + (D_X D_V X - D_{D_X V}X) + D_{D_V X}X\\ =& R(X,V)X + D^2_{X,V}X + D_{D_V X}X\\ =& R(X,V)X + (D_X S)(V) + S^2(V).\end{aligned}$$

(C) We have to compute the trace of each one of the terms in (b).
- Clearly, $\mathrm{tr}(V \mapsto R(X,V)X)$=$\mathrm{Ric}(X,X)$.
- Next, since the tensor $v \mapsto D_V X = S(v)$ is assumed to be symmetric, we have

$$|DX|^2 = \mathrm{tr}(DX \circ DX) = \mathrm{tr}(S^2).$$

- For the last tensor, we shall prove something slightly more general. Namely,

Lemma 4. *For any $(1,1)$-tensor S and vector field X, we have*

$$\mathrm{tr}(D_X S) = D_X \mathrm{tr} S.$$

From this we obtain

$$D_X \mathrm{tr} S = D_X \mathrm{div}\, X = g(X, \nabla \mathrm{div}\, X).$$

To prove Lemma 4, we assume first the truth of Lemma 5 below and then calculate

$$\begin{aligned}\mathrm{tr}(D_X S) =& \sum g((D_X S)(E_i), E_i)\\ =& \sum g(D_X(S(E_i)), E_i) - \sum g(S(D_X E_i), E_i)\\ =& D_X g(S(E_i), E_i) - \sum g(S(E_i), D_X E_i) - \sum g(S(D_X E_i), E_i)\\ =& D_X g(S(E_i), E_i), \quad \text{by Lemma 5,}\\ =& D_X \mathrm{tr} S.\end{aligned}$$

Lemma 5. *For any $(1,1)$-tensor S and vector field X, we have*

$$\sum g(S(E_i), D_X E_i) + \sum g(S(D_X E_i), E_i) = 0, \tag{1}$$

for any orthonormal frame $\{E_i\}$.

Proof of Lemma. We observe that, since $\{E_i\}$ is orthonormal, $g(E_i, E_j) \equiv 1$, and hence

$$\begin{aligned}0 =& D_X g(E_i, E_j)\\ =& g(D_X E_i, E_j) + g(E_i, D_X E_j).\end{aligned} \tag{3}$$

Now if S^* denote the adjoint to S, we have

$$
\begin{aligned}
\sum g(S(E_i), D_X E_i) + \sum g(S(D_X E_i), E_i)
&= \sum g(S(E_i), D_X E_i) + \sum g(D_X E_i, S^*(E_i)) \\
&= \sum g((S+S^*)(E_i), D_X E_i) \\
&= \sum_{i,j} g(g(S+S^*)(E_i), E_j)E_j, D_X E_i) \\
&= \sum_{i,j} g((S+S^*)(E_i), E_j) g(E_j, D_X E_i) \\
&= \sum_{i<j} g((S+S^*)(E_i), E_j) g(E_j, D_X E_i) \\
&\quad + \sum_{i>j} g((S+S^*)(E_i), E_j) g(E_j, D_X E_i) \quad (\because g(E_i, D_X E_i) = 0), \\
&= \sum_{i<j} g((S+S^*)(E_i), E_j) g(E_j, D_X E_i) \\
&\quad + \sum_{i<j} g((S+S^*)(E_j), E_i) g(E_i, D_X E_j) \\
&= \sum_{i<j} g((S+S^*)(E_i), E_j) g(E_j, D_X E_i) \\
&\quad - \sum_{i<j} g((S+S^*)(E_i), E_j) g(E_j, D_X E_i), \quad \text{by (3)} \\
&= 0. \qquad \square
\end{aligned}
$$

Theorem 6 (Bochner 1964). *If (M, g) is compact, oriented, and has* $\mathrm{Ric} \geq 0$, *then every harmonic* 1*-form is parallel.*

Proof. Suppose ω is a harmonic 1-form, X the dual vector field, and $f = \frac{1}{2} g(\omega, \omega) = \frac{1}{2}|X|^2$. Then

$$\Delta f = |DX|^2 + \mathrm{Ric}(X, X), \quad (\because \operatorname{div} X = 0).$$

$$
\begin{aligned}
\therefore 0 &= \int_M \Delta f \cdot dv_g \\
&= \int_M (|DX|^2 + \mathrm{Ric}(X, X)) \cdot dv_g \\
&\geq \int_M |DX|^2 \cdot dv_g \geq 0.
\end{aligned}
$$

We can therefore conclude that $|DX| = 0$. $\square$

Corollary 7. *If (M, g) is compact, oriented, has* $\mathrm{Ric} \geq 0$, *and has positive Ricci curvature at one point, then all harmonic* 1*-form vanish everywhere.*

Proof. From the proof of Theorem 6, we obtain $\mathrm{Ric}(X, X) \equiv 0$, under the assumption that (M, g) is compact, oriented, and has $\mathrm{Ric} \geq 0$. If the Ricci tensor is positive on T_pM, we must have $X(p) = 0$.
But then $X \equiv 0$, since X is parallel. □

Corollary 8. *If (M, g) is compact, oriented, and has* $\mathrm{Ric} \geq 0$, *then*

$$dim\, \mathbb{H}^1(M) \leq n.$$

Proof. All the harmonic 1-forms are parallel, and hence is determined by its value at one point. Therefore the linear map $\mathbb{H}^1(M) \to T_pM$ defined by $\omega \mapsto X(p)$ is injective. □

Corollary 9. *If (M, g) is compact, oriented, and has* $\mathrm{Ric} \geq 0$, *then*

$$b_1(M) \leq n = \dim M, \tag{4}$$

with equality holding iff (M, g) is a flat torus.

Proof. We know from the Hodge theorm that

$$b_1(M) = \dim \mathbb{H}^1(M).$$

Thus (4) follows from Corollary 8.

- If equality holds, we obviously have n linearly independent parallel fields E_i, $i = 1, \cdots, n$.
  This clearly implies that (M, g) is flat.
  Thus the universal covering is ($\mathbb{R}^n$,can) with $\Gamma = \pi_1(M)$ acting by isometries.
  To complete the proof, it suffices to
  Claim: (i) Γ **consistes entirely of translations;** i.e. Γ is finitely generated, abelian, with no fixed point, and hence must be $\mathbb{Z}^q$ for some q.
  (ii) $q = n$.

(i) Pull the vector fields E_i back to $\widetilde{E}_i$, $i = 1, \cdots, n$, on $\mathbb{R}^n$.
These vector fields $\widetilde{E}_i$ are again parallel and are therefore constant vector fields.
– In addtion, they are invariant under the action of Γ; i.e., for any $\gamma \in \Gamma$, we have

$$D\gamma(\widetilde{E}_i(p)) = \widetilde{E}_i(\gamma(p)), \quad i = 1, \cdots, n.$$

A basis of constanr vector fields can be invariant only under translations.

(ii) If $q < n$, then $\mathbb{Z}^q$ generates a subspace V of $\mathbb{R}^n$ with dimension $< n$.
Let $W = v^{\perp}$, the orthogonal complement of V in $\mathbb{R}^n$.
Then

$$M = \mathbb{R}^n/\mathbb{Z}^q = (V \oplus W)/\mathbb{Z}^q = (v/\mathbb{Z}^q) \oplus W,$$

which is not compact and is a contradiction. □