Left-invariant and Bi-invariant Metrics

- Since a Lie group G is a smooth manifold as well as a group, it is customary to use Riemannian metrics that link the geometry of G with the group structure.

Definition. *A Riemannian metric on a Lie group G is called* **left-invariant** *if*

(1) $$\langle u, v\rangle_x = \langle (L_a)_* u, (L_a)_* v\rangle \quad \forall a, x \in G, \quad u, v \in T_x G.$$

Similarly, a Riemannian metric is **right-invariant** *if each $R_a : G \to G$ is an isometry.*

- Since the tangent space at any point can be translated to the tangent space at the identity element of the group, the above relation (1) for left-invariance can be simply written as
$$\langle u, v\rangle = \langle (L_a)_* u, (L_a)_* v\rangle.$$

Proposition 1. *There is a one-to-one correspondence between*
(a) left-invariant metrics on a Lie group G and
(b) scalar products on the Lie algebra Lie(G)
(c) a scalar product on T_eG under the canonical isomorphism $\mathrm{Lie}(G) \ni X \mapsto X_e$.

Proof. (i) Let $\langle\ ,\ \rangle$ be a left-invariant metric on G, and let X,$Y \in \mathrm{Lie}(G)$. Then the function
$$\langle X, Y\rangle(\cdot) : G \to \mathbb{R}$$
is constant on G. Indeed, because of the left-invariance of the vector fields X, Y as well as of the metric, we have that for any $a \in G$,
$$\begin{aligned}\langle X, Y\rangle(a) =& \langle X_a, Y_a\rangle = \langle (L_a)_* X_e, (L_a)_* Y_e\rangle \\ =& \langle X_e, Y_e\rangle = \langle X, Y\rangle_e.\end{aligned}$$

Thus $\langle X, Y\rangle$ defines a scalar product on Lie(G).
(ii) Conversely, if $\langle\ ,\ \rangle_e$ is a scalar product on Lie(G), then the metric defined by
$$\langle X, Y\rangle_a = \langle (L_{a^{-1}})_* X, (L_{a^{-1}})_* Y\rangle_e \quad \forall a \in G,\ X, Y \in T_a G,$$
is a left-invariant metric on G. □

Definition. *A metric on G that is both left-invariant and right-invariant is called* **bi-invariant**.

Theorem 2. *A compact Lie group possesses a bi-invariant metric.*

- For the case of bi-invariant metrics, Proposition 1 extends as follows.

Proposition 3. *There is a one-to-one correspondence between*
(a) left-invariant metrics on a Lie group G and
(b) Ad-invariant scalar products on the Lie algebra $\mathrm{Lie}(G)$, *that is,*

$$\langle \mathrm{Ad}X, \mathrm{Ad}Y\rangle = \langle X, Y\rangle \quad \forall g \in G,\ X, Y \in \mathrm{Lie}(G). \tag{2}$$

Furthermore, the last condition is equivalent to the relation

$$\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle. \tag{3}$$

Proof. We know that

$$\mathrm{Ad}(g)X = dR_{a^{-1}}X$$

and hence, by using the **right invariance**,

$$\langle \mathrm{Ad}X, \mathrm{Ad}Y\rangle = \langle (R_{a^{-1}})_*X, (R_{a^{-1}})_*Y\rangle = \langle X, Y\rangle.$$

To show (3), let $\exp(tX)$ be the flow of X. Then

$$\begin{aligned}\langle [X, Y], Z\rangle =& \langle \mathrm{ad}_X Y, Z\rangle = \langle \frac{d}{dt}\mathrm{Ad}(\exp tX)Y\Big|_{t=0}, Z\rangle \\ =& \frac{d}{dt}\langle \mathrm{Ad}(\exp tX)Y, Z\rangle\Big|_{t=0} = \frac{d}{dt}\langle Y, \mathrm{Ad}(\exp -tX)Z\rangle\Big|_{t=0}, \quad \text{by (2)} \\ =& \langle Y, -\mathrm{ad}_X Z\rangle = -\langle Y, [X, Z]\rangle.\end{aligned}$$

Hence, $\langle [Y, X], Z\rangle = -\langle X, [Y, Z]\rangle$, which is (3).

- Conversely, if we assume (3), then, for all X, Y, $V \in \mathrm{Lie}(G)$,

$$\begin{aligned}&\langle \mathrm{Ad}(\exp tV)X, \mathrm{Ad}(\exp tV)Y\rangle \\ &= \langle \exp(\mathrm{ad}(tV)X), \exp(\mathrm{ad}(tV)Y)\rangle \\ &= \sum_i \langle \exp(\mathrm{ad}(tV)X, \frac{t^i}{i!}(\mathrm{ad}(V))^i Y\rangle \\ &= \sum_i \langle (-1)^i \frac{t^i}{i!}(\mathrm{ad}(V))^i \exp(\mathrm{ad}(tV)X, Y\rangle, \quad \text{by (3)}, \\ &= \langle \exp(\mathrm{ad}(-tV)) \exp(\mathrm{ad}(tV))X, Y\rangle = \langle X, Y\rangle,\end{aligned}$$

whenever $\exp(\pm tV)$ is defined. □

Example. The Killing form is Ad-invariant. Hence, if the Lie group G is compact and semisimple, the (negative) of the Killing form provides a bi-invariant Riemannian metric.

Geometric Aspects of a Compact Lie Group

Here we will examine various geometric quantities on a Lie goup G with a left-invariant or bi-invariant metrics.

Notation: We use the notation A^* to denote the adjoint of the linear transformation A with respect to a given inner product.

Proposition 7. *Let $\langle , \rangle$ be a left invariant metric on G, and let X, Y, Z be left invariant vector fields. Then*
(a) $D_X Y = \frac{1}{2}\{[X,Y] - (ad(X))^*(Y) - (ad(Y))^*(X)\}$;
(b) 1-parameter subgroups are geodesics iff for all X, $(ad(X))^*(X) = 0$.
(c) $\langle R(Y,X)Z, W\rangle = \langle D_X Z, D_Y W\rangle - \langle D_Y Z, D_X W\rangle - \langle D_{[X,Y]} Z, W\rangle$;
(d) $\langle R(Y,X)Y, X\rangle = \|(ad(X))^*(Y) + (ad(Y))^*(X)\|^2 - \langle (ad(X))^*(X), (ad(Y))^*(Y)\rangle - \frac{3}{4}\|[X,Y]\|^2 - \frac{1}{2}\langle [[X,Y],Y], X\rangle - \langle [[Y,X],X], Y\rangle$;

Proof. (a) By left invariance, we have

$$0 = X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_Z X\rangle - \langle Y, [Z,X]\rangle$$
$$0 = Y\langle Z, X\rangle = \langle D_Y Z, X\rangle + \langle Z, D_Y X\rangle = \langle D_Y Z, X\rangle + \langle Z, D_X Y\rangle - \langle Z, [X,Y]\rangle$$
$$0 = Z\langle X, Y\rangle = \langle D_Z X, Y\rangle + \langle X, D_Z Y\rangle = \langle D_Z X, Y\rangle + \langle X, D_Y Z\rangle - \langle X, [Y,Z]\rangle$$

Therefore

$$\begin{aligned}\langle D_X Y, Z\rangle =& \frac{1}{2}(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle)\\ &+ \frac{1}{2}(\langle Z,[X,Y]\rangle - \langle Y,[X,Z]\rangle - \langle X,[Y,Z]\rangle),\\ =& \frac{1}{2}(\langle [X,Y],Z\rangle - \langle Y,[X,Z]\rangle - \langle X,[Y,Z]\rangle).\end{aligned}$$

(b) is immediate from (a).

(c) By the left invariance, $X\langle D_Y Z, W\rangle = 0$. Therefore

$$\begin{aligned}\langle D_X D_Y Z, W\rangle =& -\langle D_Y Z, D_X W\rangle\\ -\langle D_Y D_X Z, W\rangle =& \langle D_X Z, D_Y W\rangle.\end{aligned}$$

(d) follows immediately from (a) and (c). Indeed, by (c), we have

$$\langle R(X,Y)Y, X\rangle = \langle D_X Y, D_Y X\rangle - \langle D_Y Y, D_X X\rangle - \langle D_{[X,Y]} Y, X\rangle,$$

in which, by (1), we have

$$\begin{aligned}&\langle D_X Y, D_Y X\rangle\\ =& \frac{1}{4}\langle [X,Y] - (\mathrm{ad}(X))^*(Y) - (\mathrm{ad}(Y))^*(X), [Y,X] - (\mathrm{ad}(Y))^*(X) - (\mathrm{ad}(Y))^*(X)\rangle\\ =& -\frac{1}{4}\|[X,Y]\|^2 + \frac{1}{4}\|(\mathrm{ad}(X))^*(Y) + (\mathrm{ad}(Y))^*(X)\|^2,\\ &\langle D_Y Y, D_X X\rangle = \langle (\mathrm{ad}(X))^*(X), (\mathrm{ad}(Y))^*(Y)\rangle\\ &\langle D_{[X,Y]} Y, X\rangle = \frac{1}{2}\langle [[X,Y],Y], X\rangle - \langle \mathrm{ad}([X,Y]))^*(Y), X\rangle - (\mathrm{ad}(Y))^*([X,Y]), X\rangle\\ &\qquad = \frac{1}{2}\langle [[X,Y],Y], X\rangle - \frac{1}{2}\langle [[X,Y],X], Y\rangle - \frac{1}{2}\langle [X,Y],[Y,X]\rangle\\ &\qquad = \frac{1}{2}\langle [[X,Y],Y], X\rangle + \frac{1}{2}\langle [[Y,X],X], Y\rangle + \frac{1}{2}\|[X,Y]\|^2..\end{aligned}$$

□

Corollary 8. *Let $\langle\ ,\ \rangle$ be a bi-invariant metric on G. Then*
(a) $D_X Y = \frac{1}{2}[X,Y]$;
(b) 1-parameter subgroups are geodesics.
(c) $\langle R(Y,X)Z,W\rangle = \frac{1}{4}\langle [X,W],[Y,Z]\rangle - \frac{1}{4}\langle [X,Z],[Y,W]\rangle$;
(d) $\langle R(Y,X)Y,X\rangle = \frac{1}{4}\|[X,Y]\|^2$.
In particular, the sectional curvature is nonnegative.

Proof. (a) now follows from the proof of Proposition 7.
(b) 1-parameter subgroups are orbits of left invariant vector fields, so (a)⇒(b).
(c) Substituing (a) into (c) of Proposition 7 gives

$$\langle R(Y,X)Z,W\rangle = \frac{1}{4}\langle [X,Z],[Y,W]\rangle - \frac{1}{4}\langle [Y,Z],[X,W] + \frac{1}{2}\langle [[X,Y],Z],W\rangle.\rangle \tag{4}$$

Since $\langle\ ,\ \rangle$ is bi-invariant, by (3), we have

$$\langle Y,[X,Z]\rangle = -\langle Y,[Z,X]\rangle = \langle [Z,Y],X\rangle. \tag{5}$$

Using the Jacobi identity and (5), the last term in (4) can be written as

$$\begin{aligned} -\frac{1}{2}\langle [[X,Y],Z],W\rangle &= -\frac{1}{2}\langle [Y,[Z,X]],W\rangle - \frac{1}{2}\langle [X,[Y,Z]],W\rangle \\ &= -\frac{1}{2}\langle [Z,X],[Y,W]\rangle + \frac{1}{2}\langle [Y,Z],[X,W]\rangle. \end{aligned}$$

(d) follows immediately from (c). □

Corollary 8*. *Let $\langle\ ,\ \rangle$ be a bi-invariant metric on G. Then*
(c)* $R(Y,X)Z = \frac{1}{4}\langle [[X,Y],Z]\rangle$.

Proof. By (a),

$$\begin{aligned} R(X,Y)Z =& D_Y D_X Z - D_X D_Y Z + D_{[X,Y]}Z \\ =& \frac{1}{4}[Y,[X,Z]] - \frac{1}{4}[X,[Y,Z]] + \frac{1}{2}[[X,Y],Z]. \end{aligned}$$

By the Jacobi identity $[X,[Y,Z]] + [Y,[Z,X]] = -[Z,[X,Y]]$. Hence

$$[Y,[X,Z]] - [X,[Y,Z]] = [Z,[X,Y]],$$

from which the result is obtained. □

Proposition 9. *Let G be a Lie group with a bi-invariant metric. Then for any X, Y, $Z \in Lie(G)$*
(a) The sectional curvature is given by

$$\mathcal{K}(X,Y) = \frac{1}{4}\frac{\langle [X,Y],[X,Y]\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$

(b) The Ricci curvature is given by

$$\mathrm{Ric}(X,Y) = \frac{1}{4}\langle [X,E_i],[Y,E_i]\rangle,$$

where $\{E_i\}$ is an orthonormal basis for Lie(G).

(c) If G is compact and the bi-invariant metric is the metric coming from the Killing form, then the scalar curvature is given by

$$S = \frac{1}{4}\dim(G).$$

Proof. (a) By the Ad-invariance of the inner product on Lie(G) and Corollary 8*, we have

$$\langle R(Y,X)Y,X\rangle = \frac{1}{4}\langle [[X,Y],X],Y]\rangle = \frac{1}{4}\langle [X,Y],[X,Y]\rangle.$$

(b) We compute

$$\begin{aligned}\mathrm{Ric}(X,Y) =& \mathrm{tr}\{Z \mapsto R(X,Z)Y\} = \sum_i \langle R(X,E_i)Y,E_i\rangle \\ =& \frac{1}{4}\sum_i \langle [[X,E_i],Y],E_i\rangle = \frac{1}{4}\sum_i \langle [X,E_i],[Y,E_i]\rangle,\end{aligned}$$

where we used the Ad-invariance in the last equality.

(c) We compute

$$\begin{aligned}S =& \frac{1}{4}\sum_{i,j}\langle [E_i,E_j],[E_i,E_j]\rangle = \frac{1}{4}\sum_{i,j}\langle E_i,[E_j,[E_i,E_j]]\rangle \\ =& -\frac{1}{4}\sum_{i,j}\langle E_i,[E_j,[E_j,E_i]]\rangle = -\frac{1}{4}\sum_i B(E_i,E_i) = \frac{1}{4}\dim G.\end{aligned}$$

Definition. *A Riemannian manifold (M,g) is called an* **Einstein manifold** *if the Ricci curvature tensor satisfies the equation*

$$\mathrm{Ric}(X,Y) = cg(X,Y)$$

for some constant c.

Proposition 10. *If G is semisimple and compact, and the bi-invariant metric is the metric coming from the Killing form, then*

$$\mathrm{Ric}(X,Y) = -\frac{1}{4}B(X,Y).$$

Thus G is an **Einstein manifold** *with respect to the Killing form metric.*

Proof. By Proposition 9 and the definition of the Killing form, we have

$$\begin{aligned}B(X,Y) =& \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y) = -\sum_i \langle [X,[Y,E_i]],E_i\rangle \\ =& -\sum_i \langle [Y,E_i],[X,E_i]\rangle = -4\mathrm{Ric}(X,Y). \qquad \square\end{aligned}$$