Connections in Vector Bundles and Curvature

Definition 1. *By a* **connection** *in a vector bundle $\pi : E \to M$ over a C^∞ manifold M, we mean a bilinear map*

$$D : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

satisfying the following conditions:

(i) $$D_{fX}s = fD_X s,$$

and

(ii) $$D_X(fs) = fD_X s + (Xf)s,$$

where $f \in C^\infty(M)$, $X \in \Gamma(TM)$, and $s \in \Gamma(E)$.
We call $D_X s$ the **covariant derivative** *of s relative to X.*

- We shall see that any vector bundle admits a connection.

(i) First consider the product bundle $M \times \mathbb{R}^n$.
Let $x_1, \cdots, x_n$ be the canonical coordinates in $\mathbb{R}^n$.
We take a frame field $(s_1, \cdots, s_n)$, where $s_i(p) = \frac{\partial}{\partial x_i}$, and set

$$D_X s_i = 0, \quad i = 1, \cdots, n.$$

For any $s = \sum_i a_i s_i$ and every vector field X, we set

$$D_X s = \sum_{i=1}^n (X a_i) s_i.$$

For this connection, $D_X s$ is just the partial derivative in the direction of X if s is considered as a $\mathbb{R}^n$-valued function on M.
We call it a **trivial connection** in the product bundle.

(ii) For an arbitrary vector bundle $\pi : E \to M$, we take a locally finite open covering $\{U_\alpha\}_{\alpha \in A}$ such that $\pi^{-1}(U_\alpha)$ is trivial and denote by D^α a trivial connection in each $\pi^{-1}(U_\alpha)$.

– Let $\{f_\alpha\}$ be a partition of unity for the covering U_α and define

$$D_X s = \sum_\alpha f_\alpha D_X^\alpha s.$$

It is easy to verify that this defines a connection in E.

Proposition 1. *Let D_i $(1 \le i \le k)$ be k connections in a given vector bundle. Then every linear combination $\sum_{i=1}^k D_i$, where $t_1 + \cdots + t_k = 1$, is a connection.*

Definition. *Let D be a connection in a vector bundle $\pi : E \to M$. Then the map that assigns to a pair of vector fields X, Y the operator*

$$R(X, Y) \to \frac{1}{2}(D_X D_Y - D_Y D_X - D_{[X,Y]})$$

is called the **curvature** *of the connection.*

Connection Form and Curvature Form

- Let D be a connection in a vector bundle $\pi : E \to M$, and R its curvature. In this subsection, we discuss how we can locally represent D and R by differential forms.
- Suppose we take a frame field $s_1, \cdots, s_n \in \Gamma(E_U)$, where U is a certain open subset of M. For any vector field X on U, we may write down

$$D_X s_j = \sum_{i=1}^{n} w_j^i(X) s_j \quad \text{with } \omega_j^i \in C^\infty(U).$$

— Since $\omega_j^i(fX) = f\omega_j^i(X)$, it follows that **each** ω_j^i **is a** 1**-form on** U.
In fact, these n^2 1-forms contain all the information on the connection D on U.

Definition. *Denoting them collectively as*

$$\omega = (\omega_j^i),$$

we call ω *the* **connection form** *of* D *on* U*.*

We may consider ω as a 1-form on U with values in the set $\mathrm{M}(n, \mathbb{R})$ of all $n \times n$ matrices.

- We may look at the curvature R from the same point of view.
- For any vector fields X, Y on U, we define $\Omega_j^i(X, Y) \in C^\infty(U)$ by writing

$$R(X, Y)(s_j) = \sum_{j=1}^{n} \Omega_j^i(X, Y) s_i.$$

We have
$$\Omega_j^i(Y, X) = -\Omega_j^i(X, Y), \quad \Omega_j^i(fX, gY) = fg\Omega_j^i(X, Y).$$

Hence, **each** Ω_j^i **is a** 2**-form on** U.

Definition. *Denoting them collectively as*

$$\Omega = (\Omega_j^i),$$

on U *with values in* $\mathrm{M}(n, \mathbb{R})$), *we call* Ω *the* **curvature form**.

- The following theorem describes the relationship between the connection form and the curvature form, and is called the **structure equation**.

Theorem 2. *For a vector bundle the connection form* $\omega = (\omega_j^i)$ *and the curvature form* (Ω_j^i) *are related by*
$$d\omega = -\omega \wedge \omega - \Omega.$$

Componentwise, this is
$$d\omega_j^i = -\sum_{k=1}^{n} \omega_k^i \wedge \omega_j^k - \Omega_j^i.$$

Proof. From the definition of the curvature form, we obtain

$$2R(X,Y)(s_j) = 2\sum_{i=1}^{n} \Omega^i_j(X,Y)s_j, \tag{1}$$

On the other hand, the definition of curvature gives us

(2)
$$\begin{aligned}
2R(X,Y)(s_j) &= (D_Y D_X - D_X D_Y + D_{[X,Y]})s_j \\
&= D_Y(\sum_{i=1}^{n} \omega^i_i j(X)s_i) - D_X(\sum_{i=1}^{n} \omega^i_j(Y)s_i) + \sum_{i=1}^{n} \omega^i_j([X,Y])s_i \\
&= \sum_{i=1}^{n}(Y\omega^i_j(X))s_i + \sum_{i,k=1}^{n} \omega^k_j(X)\omega^i_k(Y)s_i \\
&\quad - \sum_{i=1}^{n}(X\omega^i_j(Y))s_i - \sum_{i,k=1}^{n} \omega^k_j(Y)\omega^i_k(X)s_i + \sum_{i=1}^{n} \omega^i_j([X,Y])s_i.
\end{aligned}$$

Now if we substitute

$$\begin{aligned}
2d\omega^i_j(X,Y) =& X\omega^i_j(Y) - Y\omega^i_j(X) - \omega^i_j([X,Y]) \\
2\omega^i_k \wedge \omega^k_j(X,Y) =& \omega^i_k(X)\omega^k_j(Y) - \omega^i_k(Y)\omega^k_j(X),
\end{aligned}$$

in (2), we obtain

$$2R(X,Y)(s_j) = 2\sum_{i=1}^{n}[\sum_{k=1}^{n} \omega^i_k \wedge \omega^k_j(X,Y) - d\omega^i_j(X,Y)]s_i. \tag{3}$$

Comparing (1) and (3), we obtain the structure equation. □

Transformation Rules of The Local Expressions for A Connection And Its Curvature

- Let D be a connection in a vector bundle $\pi : E \to M$. Given two open subsets U_α and U_β in M and trivilizations

$$\begin{aligned}
\varphi_\alpha : \pi^{-1}(U_\alpha) \cong& U_\alpha \times \mathbb{R}^n, \\
\varphi_\beta : \pi^{-1}(U_\beta) \cong& U_\beta \times \mathbb{R}^n.
\end{aligned}$$

  let $g :\to \mathrm{GL}(n,\mathbb{R})$ be the transition function. We denote by ω_α, Ω_α; ω_β, Ω_β the connection and curvature forms on U_α and U_β relative to the frame fields induced by φ_α and φ_β.

Proposition 3. *We have the transformation rules*

(i) $$\omega_\beta = g_{\alpha\beta}^{-1}\omega_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta},$$

(ii) $$\Omega_\beta = g_{\alpha\beta}^{-1}\Omega_\alpha g_{\alpha\beta}.$$

Proof. Let $s_1, \cdots, s_n$ be the frame field on U_α induced by φ_α, and $t_1, \cdots, t_n$ the field induced on U_β by φ_β. On $U_\alpha \cap U_\beta$ we have

$$t_j = \sum_{i=1}^{n} g_j^i s_i, \tag{4}$$

where the components of $g_{\alpha\beta}$ are denoed by g_j^i.

– Applying D_X to (4), we obtain

$$\sum_{k=1}^{n} \omega(\beta)_j^k(X) t_k = \sum_{i=1}^{n} dg_j^i(X) s_i + \sum_{i,k=1}^{n} g_j^k \omega(\alpha)_k^i(X) s_i \tag{5}$$

where the components of ω_α and ω_β are denoted by $\omega(\alpha)_j^i$ and $\omega(\beta)_j^i$.

– Substituting (4) into (5), we obtain

$$\sum_{k=1}^{n} \omega(\beta)_j^k(X) g_k^i s_i = \sum_{i=1}^{n} dg_j^i(X) s_i + \sum_{i,k=1}^{n} g_j^k \omega(\alpha)_k^i(X) s_i.$$

Comparing the coefficients of s_i, we obtain

$$\sum_{k=1}^{n} \omega(\beta)_j^k(X) g_k^i = \sum_{i=1}^{n} dg_j^i(X) + \sum_{i,k=1}^{n} g_j^k \omega(\alpha)_k^i(X).$$

Since this holds for arbitrary X and i, j, we can write

$$g_{\alpha\beta}\omega_\beta = dg_{\alpha\beta} + \omega_\alpha g_{\alpha\beta}.$$

Multiplying by $g_{\alpha\beta}^{-1}$ on both sides, we obtain the desired equation.

(ii) From Theorem 2 we have

$$\Omega_\beta = -d\omega_\beta - \omega_\beta \wedge \omega_\beta.$$

– Now we take the exterior derivative of each side of (i).
Here functions and 1-forms appear as matrices, but their exterior derivatives can be easily handled by usual relues.
For instance, if we write g for $g_{\alpha,\beta}$ for simplicity, then exterior differentiation of $g^{-1}g = I$, where I is the unit matrix, gives rise to $dg^{-1}g + g^{-1}dg = 0$, from which we obtain

$$dg^{-1} = -g^{-1}dgg^{-1}.$$

Now from (i), we obtain

$$\begin{aligned} -\Omega_\beta =& - d\omega_\beta - \omega_\beta \wedge \omega_\beta \\ =& - g^{-1}dgg^{-1} \wedge \omega_\alpha g + g^{-1}d\omega_\alpha g - g^{-1}\omega_\alpha \wedge dg - g^{-1}dgg^{-1} \wedge dg \\ &+ (g^{-1}\omega_\alpha g + g^{-1}dg) \wedge (g^{-1}\omega_\alpha g + g^{-1}dg) \\ =& g^{-1}(d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha)g = -g^{-1}\Omega_\alpha g. \qquad \square \end{aligned}$$