Weitzenbock Formulas

- Let $E \to M$ be a Hermitian vector bundle with a metric connection D.
  Suppose E is a $Cl(M)$-module and D is a Clifford connection.

– If we consider the Dirac-type operator

$$\mathfrak{D} : C^\infty(M, E) \to C^\infty(M, E)$$

and the covariant derivative $D : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$,
then $\mathfrak{D}^2$ and D^*D are operators on $C^\infty(M, E)$ with the same principal symbol.

– It is of interset to examine their difference,
which is clearly a differential operator of order ≤ 1.
In fact, **the difference has order 0.**
This can be seen in principle from the following considerations.

(i) We have

$$\mathfrak{D}^2(f\varphi) = f\mathfrak{D}^2\varphi - 2D_{\mathrm{grad}\, f}\,\varphi - (\Delta f)\varphi,$$

where $\varphi \in C^\infty(M, E)$ and f is a scalar function.

(ii) Similarly, we compute $D^*D(f\varphi)$. The derivation property of D implies

$$D(f\varphi) = fD\varphi + df \otimes \varphi. \tag{2}$$

To apply D^* to this, first a short calculation gives

$$D^*f(u \otimes \varphi) = fD^*(u \otimes \varphi) - \langle df, u\rangle\varphi,$$

for $u \in C^\infty(M, T^*)$, $\varphi \in C^\infty(M, E)$, and hence

$$D^*(fD\varphi) = fD^*D\varphi - D_{\mathrm{grad}\, f}\varphi.$$

This gives D^* applied to the first term on the right side of (2).

– To apply D^* to the other term, we can use the identity

$$D^*(u \otimes \varphi) = -D_U\varphi - (\mathrm{div}\, U)\varphi.$$

where U is the vector field corresponding to v via the metric on M. Hence

$$D^*(df \otimes \varphi) = -D_{\mathrm{grad}\, f}\varphi - (\Delta f)\varphi.$$

Then (6) and (4) applied to (2) gives

$$D^*D(f\varphi) = fD^*D\varphi - 2D_{\mathrm{grad}\, f}\varphi - (\Delta f)\varphi.$$

Comparing (1) and (7), we have

$$(\mathfrak{D}^2 - D^*D)(f\varphi) = f(\mathfrak{D}^2 - D^*D)\varphi,$$

which implies $\mathfrak{D}^2 - D^*D$ has order zero, hence is given by a bundle map on E.

- We now derive the Weitzenbock formula for what the difference is.

Proposition 1. *If $E \to M$ is a $Cl(M)$-module with Clifford connection and associated Dirac-type operator $\mathfrak{D}$, then, for $\varphi \in C^\infty(M,E)$*

$$\mathfrak{D}^2\varphi = D^*D\varphi - \sum_{j>k} v_k v_j \mathcal{K}(e_k,e_j)\varphi, \tag{9}$$

where $\{e_j\}$ is a local orthonormal frame of vector fields, with dual frame field $\{v_j\}$, and $\mathcal{K}$ is the sectional curvature tensor of (E,D).

Proof. Starting with $D\varphi = i\sum v_j D_{e_j}\varphi$, we obtain

$$\begin{aligned}\mathfrak{D}^2\varphi &= -\sum_{j,k} v_k D_{e_k}(v_j D_{e_j}\varphi)\\ &= -\sum_{j,k} v_k[v_j D_{e_k} D_{e_j}\varphi + (D_{e_k} v_j) D_{e_j}\varphi].\end{aligned}$$

Replace $D_{e_k}D_{e_j}$ by the Hessian, using the identity

$$D^2_{e_k,e_j} = D_{e_k}D_{e_j}\varphi - D_{D_{e_k}e_j}\varphi;$$

We obtain

$$\begin{aligned}\mathfrak{D}^2\varphi &= -\sum_{j,k} v_k v_j D^2_{e_k,e_j}\varphi\\ &\quad -\sum_{j,k} v_k[v_j D_{D_{e_k}e_j}\varphi + (D_{e_k}v_j)D_{e_j}\varphi].\end{aligned} \tag{12}$$

Let us look at each of the two double sums on the right.

(i) Using $v_j^2 = 1$ and the anticommutator property $v_k v_j = -v_j v_k$ for $k \neq j$, we see that the first double sums becomes

$$-\sum_j D_{e_j,e_j}\varphi - \sum_{j>k} v_j v_k \mathcal{K}_{e_k,e_j}\varphi$$

since the antisymmetric part of the Hessian is the curvature.
This is equal to the right side of (9).

(ii) As for the remaining double sum in (12), for any $p \in M$, we choose a local orthonrmal frame field such that $D_{e_j}e_k = 0$ at p, and then this tem vanishes at p. □

- We denote the difference $\mathfrak{D}^2 - D^*D$ by $\mathfrak{K}$, so

$$(\mathfrak{D}^2 - D^*D)\varphi = \mathfrak{K}\varphi, \quad \mathfrak{K} \in C^\infty(M, \mathrm{End}\, E).$$

The formula for $\mathfrak{K}$ in (9) can be written as

$$\mathfrak{K}\varphi = -\frac{1}{2}\sum_{j,k} v_k v_j K(e_k,e_j)\varphi \tag{15}$$

The general formula for $\mathfrak{K}$ simplifies further in some important special cases.

Proposition 2. *Let $E = \Lambda^* M$, with $Cl(M)$-module structure and connection, so $\mathfrak{K} \in C^\infty(M, End\,\Lambda^*)$. Then*

$$u \in \Lambda^1 M \Rightarrow \mathfrak{K} = \mathrm{Ric}(u). \tag{16}$$

Proof. The curvature of $\Lambda^* M$ is the sum of curvatures of each factor $\Lambda^k M$.
In particular, if $\{e_j, v_j\}$ is a local dual pair of frame fields,

$$\mathcal{K}(e_i, e_j) v_k = -R^k{}_{\ell ij} v_\ell, \tag{17}$$

where $R^k{}_{\ell ij}$ are the components of the Riemann tensor, with respect to the frame fields, and use the summation convention.
– In view of (15), the desired identity (16) will hold provided

$$\frac{1}{2} v_i v_j v_\ell R^k{}_{\ell ij} = \mathrm{Ric}(v_k), \tag{18}$$

so **it remains to establish the identity (18).**
– Since if (i, j, ℓ) are distinct, $v_i v_j v_\ell = v_\ell v_i v_j = v_j v_\ell v_i$,
and since by Bianchi's first identity

$$R^k{}_{\ell ij} + R^k{}_{j\ell i} + R^k{}_{ij\ell} = 0,$$

in summing the left side of (18), the sum over distict (i, j, ℓ) vanishes.
– Thus the only contribution arise from $i = \ell \neq j$ and $i \neq \ell = j$.
– Therefore, the left side of (18) is equal to

$$\frac{1}{2}(-v_j R^k{}_{iij} + v_i R^k{}_{jij}) = v_j R^k{}_{jij} = \mathrm{Ric}(v_k).$$

which completes the proof. □

- We next derive Lichnerowicz's calculation of $\mathfrak{K}$ when $E = S(P_{\mathrm{Spin}})$, the spinor bundle of a manifold M with spin structure.
  First we need an expression for the curvature of $S(P_{\mathrm{Spin}})$.

Lemma 3. *The curvature tensor of the spinor bundle $S(P_{\mathrm{Spin}})$ is given by*

$$\mathcal{K}(e_i, e_j)\varphi = \frac{1}{4} R^k{}_{\ell ij} v_k v_\ell \varphi. \tag{20}$$

Proposition 4. *For the spin bundle $S(P_{\mathrm{Spin}})$, $\mathfrak{K} \in C^\infty(M, End\,S(P_{\mathrm{Spin}}))$ is given by*

$$\mathfrak{K}\varphi = \frac{1}{4}(\mathrm{Scal})\varphi,$$

whose Scal *is the scalar curvature of M.*

Proof. Using (20), the general formula (15) yields

$$\varphi = -\frac{1}{8} R^k{}_{\ell ij} v_i v_j v_k v_\ell \varphi = \frac{1}{8} v_i v_j v_\ell R^k{}_{\ell ij} v_k \varphi,$$

the last identity holding by the anticommutation relations; note that only the sums over $k \neq \ell$ counts. This becomes

$$\begin{aligned}\mathfrak{K} =& \frac{1}{4} v_i v_k R^k{}_{jij} \varphi, \quad \text{by (17) and (18)} \\ =& \frac{1}{4}(\mathrm{Ric}_{ii})\varphi \\ =& \frac{1}{4}(\mathrm{Scal})\varphi. \qquad \square\end{aligned}$$

Proposition 5. *Let $E \to M$ have a metric connection D, with curvature R^E. For the twisted Dirac operator on sections of $F = S(P_{\mathrm{Spin}}) \otimes E$, the section $\mathfrak{K}$ of $\mathrm{End}\, F$ has the form*

$$\mathfrak{K}\varphi = \frac{1}{4}(\mathrm{Scal})\varphi - \frac{1}{2}\sum_{i,j} v_i v_j R^E(e_i, e_j)\varphi;$$

here $R^E(e_i, e_j)$ is shorthand for $I \otimes R^E(e_i, e_j)$ acting on $S(P_{\mathrm{Spin}}) \otimes E$.

Proof. This formula is a consequence of the general formula (15) and the argument proving Proposition 4, since the curvature of $S(P_{\mathrm{Spin}}) \otimes E$ is

$$\mathcal{K} \otimes I + I \otimes R^E,$$

$\mathcal{K}$ being the curvature of $S(P_{\mathrm{Spin}})$, given by (20). $\square$

Proposition 6. *If M is compact and connected, and the section $\mathfrak{K}$ in (14)-(15) has the property that $\mathfrak{K} \geq 0$ on M and $\mathfrak{K} > 0$ at some point, then $\ker \mathfrak{D} = 0$.*

Proof. This is immediate from $(\mathfrak{D}^2\varphi, \varphi) = (\varphi, \varphi) + \|D\varphi\|^2_{L^2}$. $\square$

Proposition 7. *If M is compact Riemannian manifold with positive Ricci tensor, then $b_1(M) = 0$, that is, the de Rham cohomology group $H^1(M, \mathbb{R}) = 0$.*

Proof. **Claim: if $u \in \Lambda^1(M)$ and $du = \delta u = 0$, then $u = 0$.**
By Proposition 2, we have

$$\|\mathfrak{D}u\|^2_{L^2} = (\mathrm{Ric}(u), u) + \|Du\|^2_{L^2}.$$

On the other hand, we have $\mathfrak{D}u = 0$, by hypothesis. $\square$

Proposition 8. *If M is a compact, connected Riemannian manifold with a spin structure whose scalar curvature is ≥ 0 on M and > 0 at some point, then M has no nonzero harmonic spinors, that is, $\ker \mathfrak{D} = 0$ in $C^\infty(M, S(\widetilde{P}))$.*

Proof. In view of (21), this is a special case of Proposition 6. $\square$