**The Tangent Space of a Lie Group – Lie Algebras**

- We will see that it is possible to associate to every point of a Lie group $G$ a real vector space, which is the tangent space of the Lie group at that point.
- – By use of certain diffeomorphisms on the Lie group, namely left or right translations, we will see that it is enough to study the tangent space of a Lie group of a Lie group at the identity element $e$.
- – The tangent space at that point is not only a vector space but it is isomorphic to what is defined below to be the **Lie algebra** of a Lie group $G$.

- Let $a$ be an element of a Lie group $G$. We define the maps

$$\begin{aligned} L_a : G \to G, \quad L_a(g) =& ag \quad \text{(left translation)}, \\ R_a : G \to G, \quad R_a(g) =& ga \quad \text{(right translation)}, \end{aligned}$$

  These maps are smooth, in fact they are diffeomorphisms since the inverse of $L_a$ is $L_{a^{-1}}$ and the inverse of $R_a$ is $R_{a^{-1}}$.
- – They can be used in order to get around in a Lie group; nemely, any $a \in G$ can be moved to $b \in G$ by $L_{a^{-1}b}$ or $R_{ba^{-1}}$.
- – The induced map $(L_{g^{-1}})_* : T_gG \to T_eG$ is a vector space isomorphism (similarly for the right translations).

**Proposition 5.** *Any Lie group $G$ is parallelizable, i.e. $TG \cong G \times T_eG$.*

*Proof.* Let $X_g$ be the value of a vector field $X$ at a point $g \in G$. Then the map

$$X_g \mapsto (g, (L_{g^{-1}})_*(X_g)).$$

is the desired isomorphism. □

**Definition.** *A vector field $X$ on a Lie group $G$ is* **left-invariant** *if $X \circ L_a = (L_a)_*(X)$ for all $a \in G$, or more explicitly*

$$X_{ag} = (L_a)_*(X_g), \quad \forall a, g \in G.$$

- A left-invariant vector field has the important property that it is determined by its value at the identity element $e$ of the Lie group, since $X_a = (L_a)_*(X_e)$ for all $a \in G$.
- Since multiplication in $G$ is smooth, so is a left-invariant vector field.

- Let $\mathrm{Lie}(G)$ denote the set of all left-invariant vector fields on a Lie group $G$.
- – The usual addtion of vector fields and scalar multiplication by real numbers make $\mathrm{Lie}(G)$ a vector space.
- – Furthermore, $\mathrm{Lie}(G)$ is closed under the bracket operation on vector fields, i.e.

$$(L_a)_*[X, Y]_p f = [X, Y]_p f,$$

  for all $X$, $Y$: left invariant vector fields on $G$, $a, p \in G$, $f \in C^\infty(M)$.
- – Thus, $\mathrm{Lie}(G)$ is a Lie algebra, called the **Lie algebra of** $G$.

- The dimension of $\mathrm{Lie}(G)$ is equal to the dimension of $G$ because of the following:

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**Proposition 6.** *The function $X \to X_e$ defines a linear isomorphism between the vector space* Lie$(G)$ *and* $T_eG$.

*Proof.* The function is obviously linear.
– It is one-to-one, since if $X_e = 0$, then $X_g = (L_g)_*(X_e) = 0$ for all $g \in G$.
– It is onto. Indeed, for all $v \in G$, define the vector field $X^v$ by $X^v_g = (dL_g)_e(v)$ for all $g \in G$. Then $X^v$ is left-invariant and $X^v_e = v$. □

- Through this isomorphism, we can define a Lie bracket on the tangent space $T_eG$ by $[u, v] = [X^u, X^v]_e = [(dL_g)_e(u), (dL_g)_e(v)]$.

**Example 1.** The set $M(n, \mathbb{K})$ of all real matrices is a Lie algebra if we set

$$[A, B] = AB - BA.$$

**Example 2.** $\mathrm{Lie}(\mathrm{GL}(n, \mathbb{K})) \cong M(n, \mathbb{K})$. Indeed, recall that $\mathrm{GL}(n, \mathbb{K})$ is (canonically isomorphic) to $\mathrm{M}(n, \mathbb{K})$. Hence we obtain the following canonical vector space isomorphisms:

$$\mathrm{Lie}(\mathrm{GL}(n, \mathbb{K}) \cong T_e(\mathrm{GL}(n, \mathbb{K})) \cong T_e(M(n, \mathbb{K})) \cong M(n, \mathbb{K}),$$

where $e$ is the $n \times n$ identity matrix.
– The first isomorphism is obtained from Proposition 6,
the second is the open submanifold identification,
and the third one is the canonical vector space identification.
– By a straightforward coordinate calcultion we see that the brackets are also preserved.

**Example 3.** Let $V$ be a vector space of dimension $n$. Let

$$\begin{aligned}\mathrm{End}(V) &= \text{the set of all linear maps from } V \text{ to itself} \cong M(n, \mathbb{R}),\\ \mathrm{Aut}(V) &= \text{the set of all invertible linear maps from } V \text{ to itself} \cong \mathrm{GL}(n, \mathbb{K}).\end{aligned}$$

The set $\mathrm{End}(V)$ becomes a **Lie algebra** of dimension $n^2$ if we set

$$[f_1, f_2] = f_1 \circ f_2 - f_2 \circ f_1.$$

On the other hand, $\mathrm{Aut}(V)$ is a **Lie group** (it inherits a manifold structure as an open subset of $\mathrm{End}(V)$, and the group operation is the composition of maps), and

$$T_e(\mathrm{Aut}(V)) \cong \mathrm{End}(V),$$

Here, $e$ denotes the identity transformation on $V$.

**One-parameter Subgroups**

- Here we will describe a second characterization of the tangent space of a Lie group as **the set of its one-parameter subgroups**.

— This is also called the **infinitesimal** description of a Lie group.

**Definition.** *A* **one-parameter subgroup** *of a Lie group $G$ is a smooth homomorphism $\phi : (\mathbb{R}, +) \to G$.*

Thus $\phi : \mathbb{R} \to G$ is a curve such that

$$\phi(s + t) = \phi(s)\phi(t), \quad \phi(0) = e, \quad \phi(-t) = (\phi(t))^{-1}.$$

**Examples.**

(1) The map $\phi(t) = e^t$ is a 1-parameter subgroup of the additive Lie group $\mathbb{R}$.
(2) The map $\phi(t) = e^{it}$ is a one-parameter subgroup of the circle $\mathbb{S}^1 = U(1)$.
(3) The map $\phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ is a one-parameter subgroup in $U(2)$.
(4) $\phi(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^t \end{pmatrix}$ is a 1-parameter subgroup in $\mathrm{GL}(3, \mathbb{R})$.

**Theorem 7.** *The map $\phi \mapsto (\phi_0)_*(\frac{d}{dt})$ defines a one-to-one correspondence between one-parameter subgroup of $G$ and $T_eG$.*

*Proof.* Let $v \in T_eG$ and $X_g^v = (DL_g)_e(v)$ be the correspnding left-invariant vector field. We need to find a smooth homomorphism

$$\phi_v : \mathbb{R} \to G, \quad \text{which is the inverse of } \phi \mapsto d\phi_0.$$

Let $\phi : I \to G$ (I being an open interval containing 0) be the unique integral curve of $X^v$ such that $\phi(0) = e$ and $(\phi_t)_*(\frac{d}{dt}) = X^v_{\phi(t)} = (dL_{\phi(t)})_e(v)$.

– This curve is a homomorphism because if we fix an $s \in I$ such that $s + t \in I$ for all $t \in I$, then the curves

$$t \mapsto \phi(s + t) \quad \text{and} \quad t \mapsto \phi(s)\phi(t)$$

satisfy the previous equation (the second by the leftinvariance of $X^v$), and take the common value $\phi(s)$ when $t = 0$. Thus by the uniqueness of the solution we obtain that

$$\phi(s + t) = \phi(s)\phi(t), \quad \forall s, t \in I.$$

– Now extend $\phi$ to all of $\mathbb{R}$ by defining $\phi_v(t) = \phi(\frac{t}{n})^n$ and this is the desired homomorphism.

The map $v \mapsto \phi_v$ is the inverse of $\phi \mapsto (\phi_0)_*(\frac{d}{dt})$ and this completes the proof. □

**Corollary 8.** *For each $X \in$Lie$(G)$, there exists a unique one-parameter subgroups of a Lie group $\phi_X : \mathbb{R} \to G$ such that $\phi'_X(0) = X$.*

*Proof.* Use the identification of the tangent space $T_eG$ and $\mathrm{Lie}(G)$. □

**Definition.** *The* **exponential map** $\exp : Lie(G) \to G$ *is defined by*

$$\exp(X) = \phi_X(1),$$

*where $\phi_X$ is the unique one-parameter subgroup of $X$.*

**Lemma.** *$\phi_X(st) = \phi_{sX}(t)$. In paticular,*

$$\exp(tX) = \phi_{tX}(1) = \phi_X(t).$$

*Proof.* Consider the map $h(t) = \phi_X(st)$. This is a one-parameter subgroup with $h'(t) = s\phi'_X(st)$, so

$$h'(0) = s\phi'_X(0) = sX.$$

On the other hand, by Corollary 8, $\phi'_{sX}(0) = sX$, hence by uniqueness it follows that

$$\phi_X(st) = \phi_{sX}(t). \qquad \square$$

**Corollary 9.** *For $X \in Lie(G)$, the curve $\gamma(t) = \exp(tX)$ is the unique homomorphism in $G$ with $\gamma'(0) = X$.*
*Also, since $\phi_X$ is a homomorphism, it follows that*

$$\exp(s+t)X = \exp sX \cdot \exp tX \quad \text{and} \quad (\exp tX)^{-1} = \exp(-tX).$$

**Lemma.** *For $X \in Lie(G)$,*

$$(d\exp)_{0_e}(X) = X.$$

*Proof.* Take the curve $\alpha(t) = tX$ in Lie$(G)$ with $\alpha(0) = 0_p$ and $\alpha'(0) = X \in$Lie$(G)$. Then

$$(d\exp)_{0_p}(X) = \frac{d}{dt}(\exp \circ \alpha)\bigg|_{t=0} = \frac{d}{dt}(\exp(tX))\bigg|_{t=0} = X. \qquad \square$$

**Proposition 10.** *There is a neighborhood of $0_e \in$Lie$(G)$ which is mapped diffeomorphically by* $\exp$ *onto a neighborhood of $e \in G$.*

- If $\phi(t)$ is a one-parameter subgroup of $G$, then we can express its derivative as follows:

  (1) $$\phi'(t) = \lim_{h\to 0} \frac{1}{h}[\phi(t+h) - \phi(t)] = \lim_{h \to 0} \frac{1}{h}[(\phi(h) - e)\phi(t)] = A\phi(t),$$

  where $A = \lim_{h\to 0} \frac{\phi(h)-e}{h}$. ( This limit exists because the group is a manifold whose coordinates are smooth functions.)
- If $A$ is a matrix, the matrix $e^{At}$ is defined and the curve $\phi(t) = e^{tA}$ is the (unique) solution of the differentrial equation (1) with the initial condition $\phi(0) = A$. The matrix $A$ is called the **infinitesimal generator** of the subgroup $\phi(t)$.

**Example 1.**If we take the one-prameter subgroup $\phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ of $U(2)$, then

$$\phi'(t) = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

The infinitesimal generator of $\phi(t)$ is the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in the sense that $\phi(t) = e^{At}$.

**Proposition.** $\mathrm{Lie}(O(n)) = \{A \in M(n, \mathbb{R}), A^T = -A\}$ *of all* **skew-symmetric** *real matrices. Matrices obeying the condition* $A^T = -A$ *vanish on the diagonal. Hence* $\dim \mathrm{O}(n) = \frac{1}{2}n(n-1)$.

*Proof.* Let $\gamma(s)$ be a curve in $M(n, \mathbb{R})$ with $\gamma(0) = I$, that lies in $\mathrm{O}(n)$, i.e.

$$\gamma(s)^T \gamma(s) = I.$$

Differentiating at $s = 0$ we obtain that $\gamma'(0)^T = -\gamma'(0)$, thus $T_I O(n) \subset \{A \in M(n, \mathbb{R}), A^T = -A\}$. On the other hand, if $\overline{A}^T = -A$, then

$$(e^{sA})^T = (e^{sA})^{-1},$$

and then $\gamma(s) = e^{sA}$ is a curve in $M(n, \mathbb{R})$ with $\gamma(0) = I$ and $\gamma(\mathbb{R}) \subset O(n)$. Differentiating at $s = 0$ we obtain $\gamma'(0) = A \subset T_I O(n)$, so $\{A \in M(n, \mathbb{R}), A^T = -A\} \subset T_I O(n)$. $\square$

**Proposition.** *Lie*$(O(n)) \cong$*Lie*$(SO(n))$. *Hence* $\dim \mathrm{SO}(n) = \frac{1}{2}n(n-1)$.

**Proposition.** *The Lie algebra of the unitary group* $U(n)$ *is the set*

$$\mathrm{Lie}(U(n)) = \{A \in M(n, \mathbb{C}),\ \overline{A}^T = -A\}$$

*of all* **skew-hermitian** *complex matrices. The diagonal entries are pure imaginary and* $\dim \mathrm{U}(n) = n^2$.

**Proposition.** *The Lie algebra of* $SU(n)$ *and* $SL(n, \mathbb{R})$ *is the set*

$$\mathrm{Lie}(SU(n)) = \{A \in M(n, \mathbb{C}),\ \overline{A}^T = -A \text{ and } \mathrm{tr} A = 0\}$$
$$\mathrm{Lie}(SL(n)) = \{A \in M(n, \mathbb{R}),\ \mathrm{tr} A = 0\}.$$

*Hence* $\dim SU(n) = n^2 - 1$.

*Proof.* Use the relation $\det(e^{tA}) = e^{t\,\mathrm{tr} A}$. $\square$

**Proposition.** *The Lie algebra of the orthogonal group* $Sp(n)$ *is the set*

$$\begin{aligned} \mathrm{Lie}(\mathrm{Spin}(n)) =& \{A \in M(n, \mathbb{H}),\ \overline{A}^T = -A\} \\ =& \{A \in M(2n, \mathbb{C}),\ \overline{A}^T = -A \ \text{ and } \ A^T J + JA = 0\}. \end{aligned}$$