Integration on Lie Groups

Definition. *Let G be a Lie group. A covariant tensor field or differential form σ on G is said to be* **left-invariant** *if $L_g^*\sigma = \sigma$ for all $g \in G$.*

Proposition 1. *If ω is a left-invariant 1-form and X is a left-invariant vector field, then $\omega(X)$ is a constant function on G.*

- Let G be an n-dimensional Lie group. Since G is paralleizable, G is orientable. We now fix once and for all an orientation on G.

- Consider the left-invariant n-form on G.
- Since such a form is uniquely determined by its value at a point, and since the $\Lambda^n(T_gG)$, for a fixed $g \in G$, is one-dimensional, there is exactly a one-dimensional space of left-invariant n-forms on G.
- Choose a non-zero left-invariant n-form consistent with the fixed orientation on G.

Proposition 2. *Let G be a Lie group endwed with a left-invarant orientation. Then G has a left-invariant orientation form that is uniquely defined up to a constant multiple.*

Proof. Let $E_1, \cdots, E_n$ be a left-invariant global frame on G (i.e. a basis for the Lie algebra of G). By replacing E_1 with $-E_1$ if necessary, we may assume that this frame is positively oriented.
— Let $\varepsilon^1, \cdots, \varepsilon^n$ be the dual coframe.
— Left invariance of E_j implies that

$$(L_g^*\varepsilon^i)(E_j) = \varepsilon^i(L_{g_*}E_j) = \varepsilon^i(E_j),$$

which shows that $L_g^*\varepsilon^i = \varepsilon^i$, so ε^i is left-invariant.

(i) Let $\Omega = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$. Then

$$L_g^*\Omega = L_g^*\varepsilon^1 \wedge \cdots \wedge L_g^*\varepsilon^n = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n = \Omega,$$

so Ω is left-invariant as well.
- $\because \Omega(E_1, \cdots, E_n) = 1 > 0$, $\therefore \Omega$ is an orientation form for the given orientation.
- Clearly, any positive constant multiple of Ω is also a left-invariant orientation form.

(ii) Conversely, if $\widetilde{\Omega}$ is any other left-invariant orientation form, we can write

$$\widetilde{\Omega}_e = c\Omega_e, \quad \text{for some positive number } c.$$

Using left-invariance, we find that

$$\widetilde{\Omega}_g = L_{g^{-1}}^*\widetilde{\Omega}_e = cL_{g^{-1}}^*\Omega_e = c\Omega_g,$$

which proves that $\widetilde{\Omega}$ is a positive constant multiple of Ω. $\square$

Proposition 3. *Let G be a compact Lie group endwed with a left-invarant orientation. Then G has a unique left-invariant orientation form Ω with the property that $\int_G \Omega = 1$.*

Proof. Since G is compact and oriented, $\int_G \Omega$ is a positive number, and hence we can define

$$\widetilde{\Omega} = (\int_G \Omega)^{-1}\Omega.$$

Claerly, $\widetilde{\Omega}$ is the unique left-invariant orientation form for which G has unit volume. $\square$

Remark. The orientation form $\widetilde{\Omega}$ whose existence is asserted in this proposition is called the **Harr volume form** on G, and often denoted by dV.
Similarly, the map $f \mapsto \int_G f\, dV$ is called the **Harr integral**.

- Consider the diffeomorphism L_σ for $\sigma \in G$. Then since $(L_\sigma)^* dV = dV$, L_σ is orientation-preserving, so that, for $f \in C^\infty(G)$ with compact support,

  $$\int_G f = \int_G f dV = \int_G (L_\sigma)^*(f dV) = \int_G (f \circ L_\sigma) dV = \int_G f \circ L_\sigma.$$

  In other words, the integral of a smooth function f on G is the same as the integral of any of its left-tanslates $f \circ L_\sigma$.
  Accordingly, we call the the Harr integral **left-invariant**.

Question: To what extent the Harr integral is also right invariant? That is, when do we have

$$\int_G f = \int_G f \circ R_\sigma, \quad \forall \sigma \in G?$$

Lemma. *The form $(R_\sigma)^* dV$ is also left-invariant.*

Proof. $(L_\tau)^*(R_\sigma)^* dV = (R_\sigma)^*(L_\tau)^* dV = (R_\sigma)^* dV$, for all $\tau \in G$. $\square$

Corollary. *The form $(R_\sigma)^* dV$ is some constant multiple of dV.*

- Thus there is defined a function $\widetilde{\lambda} : G \to \mathbb{R}$ such that

  $$(R_\sigma)^* dV = \widetilde{\lambda}(\sigma) dV.$$

  It is easy to check that $\widetilde{\lambda}$ is C^∞. We let

  $$\lambda(\sigma) = |\widetilde{\lambda}(\sigma)|.$$

  Then, for each $\sigma \in G$,

  $$\int_G f dV = \int_G (f \circ R_\sigma) \lambda(\sigma) \omega.$$

  Thus we obtain the following.

Lemma. *The Harr integral is right invariant iff $\lambda \equiv 1$ on G.*

- Observe that

$$\lambda(\sigma\tau) = \lambda(\sigma)\lambda(\tau), \tag{1}$$

so that λ is a Lie group homomorphism of G into the multiplicative group $\mathbb{R}^+$.

Definition. *λ is called the* **modular function***.*
A lie group G for which $\lambda \equiv 1$ is called **unimodular***.*

Theorem 4. *Each compact Lie group G is unimodular.*

Proof. For each $\sigma \in G$,

$$1 = \int_G dV = \lambda(\sigma) \int_G dV = \lambda(\sigma). \qquad \square$$

Theorem 5. *The Harr integral on a compact Lie group is both left and right invariant.*

Applicaion of Theorem 5

Definition. *Let G be a Lie group and let $\alpha : G \to \mathrm{Aut}(V)$ be a representation into the automorphisms of a real or complex inner product space V. The representation α is called* **unitary** *(respectively* **orthogonal***) in the case in which V is complex (respectively, real) inner product space if*

$$\langle \alpha(\tau)v, \alpha(\tau)w \rangle = \langle v, w \rangle, \quad \forall v, w \in V \quad \text{and} \quad \forall \tau \in G. \tag{2}$$

Theorem 6. *Let G be compact and V complex (respectively real). Then there is an inner product on V with respect to which α is unitary (respectively orthogonal).*

Proof. Let $\{\ ,\ \}$ be any inner product on V. Set

$$\langle n, w \rangle = \int_G \{\alpha(\sigma)v, \alpha(\sigma)w\} d\sigma,$$

where we use $d\sigma$ to denote that we are considering the integrand as a function of σ in G.
It is immediate that $\langle\ ,\ \rangle$ is again an inner product, and

$$\begin{aligned}
\langle \alpha(\tau)v, \alpha(\tau)w \rangle &= \int_G \{\alpha(\sigma)\alpha(\tau)v, \alpha(\sigma)\alpha(\tau)w\} d\sigma \\
&= \int_G \{\alpha(\sigma\tau)v, \alpha(\sigma\tau)w\} d\sigma, \qquad \text{by (1)} \\
&= \int_G \{\alpha(\sigma)v, \alpha(\sigma)w\} d\sigma, \quad \text{by the } \textbf{right invariance} \text{ on } G \\
&= \langle v, w \rangle. \qquad \square
\end{aligned}$$