The Complex Projective Space

Definition. *Complex projective n-space, denoted by $\mathbb{CP}^n$, is defined to be the set of 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection*

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n.$$

Definition*. *A complex linear subspace of $\mathbb{C}^{n+1}$ of complex dimension one is called* **line**. *Define the complex projective space $\mathbb{CP}^n$ as the space of all lines in $\mathbb{C}^{n+1}$.*

- Thus, $\mathbb{CP}^n$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation

  $$z \sim w. \Leftrightarrow \exists \lambda \in \mathbb{C} \setminus \{0\} \ni w = \lambda z.$$

  Namely, two points of $\mathbb{C}^{n+1} \setminus \{0\}$ are equivalent iff they are complex linearly dependent, i.e. lie on the same line.
  Denote the equivalence class of z by $[z]$.

⊙ We also write

$$z = (z^0, \cdots, z^n) \in \mathbb{C}^{n+1}$$

and define

$$U_i = \{[z] : z^i \neq 0\} \subset \mathbb{CP}^n,$$

i.e. the space of all lines not contained in the complex hyperplane $\{z^i = 0\}$.

— We then obtain a bijection $\varphi_i : U_i \to \mathbb{C}^n$ via

$$\varphi_i([z^0, \cdots, z^n]) := \left(\frac{z^0}{z^i}, \cdots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \cdots, \frac{z^n}{z^i}\right).$$

Thus $\mathbb{CP}^n$ becomes a smooth manifold, because, assuming w.l.o.g. $i < j$, the transition maps

$$\varphi_j \circ \varphi_i^{-1} : \varphi(U_i \cap U_j) = \{z = (z^1, \cdots, z^n) \in \mathbb{C}^n : z^j \neq 0\} \to \varphi(U_i \cap U_j)$$

$$\begin{aligned}\varphi_j \circ \varphi_i^{-1}(z^1, \cdots, z^n) =& \varphi([z^1, \cdots, z^i, 1, z^{i+1}, \cdots, z^n]) \\ =& \left(\frac{z^1}{z^j}, \cdots, \frac{z^i}{z^j}, \frac{1}{z^j}, \frac{z^{i+1}}{z^j}, \cdots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \cdots, \frac{z^n}{z^j}\right)\end{aligned}$$

are diffeomorphisms.

- The vector space structure of $\mathbb{C}^{n+1}$ induce an analogous structure on $\mathbb{CP}^n$ by homogenization:

– Each linear inclusion $\mathbb{C}^{m+1} \subset \mathbb{C}^{n+1}$ induces an inclusion $\mathbb{CP}^m \subset \mathbb{CP}^n$.
The image of such an inclusion is called **linear subspace**.

– The image of a hyperplane in $\mathbb{C}^{n+1}$ is again called **hyperplane**,
and the image of a two-dimensional space $\mathbb{C}^2$ is called **line**.

- Instead of considering $\mathbb{CP}^n$ as a quotient of $\mathbb{C}^{n+1} \setminus \{0\}$, we may also view it as a **compactification** of $\mathbb{C}^n$.

— One says that the hyperplane H at infinity is added to $\mathbb{C}^n$; this means the following: the inclusion
$$\mathbb{C}^n \to \mathbb{CP}^n$$
is given by
$$(z^1, \cdots, z^n) \mapsto [1, z^1, \cdots, z^n].$$
Then
$$\mathbb{CP}^n \setminus \mathbb{C}^n = \{[z] = [0, z^1, \cdots, z^n]\} =: H,$$
where H is a hyperplane $\mathbb{CP}^{n-1}$. It follows that
$$\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1} = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^0. \tag{1}$$

Proposition. *$\mathbb{C}P^1$ is diffeomorphic to $\mathbb{S}^2$.*

Proof. It follows from (1) that the two spaces are homeomorphic.
In order to see that they are **diffeomorphic**, we recall that $\mathbb{S}^2$ can be described via stereographic projection from the north pole $(0, 0, 1)$ and the south pole $(0, 0, -1)$ by two charts with image $\mathbb{C}$, namely
$$\varphi_1(x^1, x^2, x^3) = \left(\frac{x^1}{1 - x^3}, \frac{x^2}{1 - x^3}\right)$$
$$\varphi_1(x^1, x^2, x^3) = \left(\frac{x^1}{1 + x^3}, \frac{x^2}{1 + x^3}\right),$$
and the transition map $z \mapsto \frac{1}{z}$. This, however, is nothing but the transition map $[1, z] \mapsto [\frac{1}{z}, 1]$ of $\mathbb{CP}^1$. □

Proposition. *The quotient map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ is smooth. The restriction of π to $\mathbb{S}^{2n+1}$ is a surjective submersion.*

⊙ Define an action of $\mathbb{S}^1$ on $\mathbb{S}^{n+1}$ by
$$z \cdot (w^1, \cdots, w^{n+1}) = (zw^1, \cdots, zw^{n+1}).$$
This action is smooth, free and proper. Thus, we have the following.

Proposition. $\mathbb{CP}^n \cong \mathbb{S}^{2n+1}/\mathbb{S}^1$.

⊙ Each line in $\mathbb{C}^{n+1}$ intersects $\mathbb{S}^{2n+1}$ in a circle $\mathbb{S}^1$, and we obtain the point of $\mathbb{CP}^n$ defined by this line by identifying all points on $\mathbb{S}^1$.

Proposition. *$\mathbb{CP}^n$ can be uniquely given the structure of smooth, compact, real $2n$-dimensional manifold on which the Lie group $U(n+1)$ acts smoothly and transitively. In other words, $\mathbb{CP}^n$ is a homogeneous $U(n+1)$-space.*

Proof. The unitary group $U(n+1)$ acts on $\mathbb{C}^{n+1}$ and transforms complex subspaces into complex subspaces, in particular lines to lines. Therefore, $U(n+1)$ acts on $\mathbb{CP}^n$. □

Proposition. *The round metric on* $\mathbb{S}^{2n+1}$ *decends to a homogeneous and isotropic Riemannian metric on* $\mathbb{CP}^{n+1}$*, called the* **Fubini-Study metric**.

- The projection
$$\pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$$
is called **Hopf map**. In particular, since $\mathbb{CP}^1 = \mathbb{S}^2$, we obtain a map
$$\pi : \mathbb{S}^3 \to \mathbb{S}^2$$
with fiber $\mathbb{S}^1$.

Hopf Fibration

We have the smooth map
$$H : \mathbb{C}^2 \setminus \{0\} \to \mathbb{S}^2$$
$$H : (u, v) \mapsto \left(\frac{|v|^2 - |u|^2}{|u|^2 + |v|^2}, \ \frac{2u\overline{v}}{|u|^2 + |v|^2} \right).$$

- On $\mathbb{S}^3(1)$, write the metric as
$$dt^2 + \sin^2(t) d\theta_1^2 + \cos^2(t) d\theta_2^2, \quad t \in [0, \pi/2],$$
and use the complex notation,
$$(t, e^{i\theta_1}, e^{i\theta_2}) \mapsto (\sin(t)e^{i\theta_1}, \cos(t)e^{i\theta_2})$$
to describe the isometric embedding
$$(0, \frac{\pi}{2}) \times \mathbb{S}^1 \times \mathbb{S}^1 \hookrightarrow \mathbb{S}^3(1) \subset \mathbb{C}^2.$$

- Since the Hopf fibers come from complex scalar multiplication, we see that they are of the form
$$\theta \mapsto (t, e^{i(\theta_1+\theta)}, e^{i(\theta_1+\theta)}).$$
- On $\mathbb{S}^2(\frac{1}{2})$ use the metric
$$dr^2 + \frac{\sin^2(2r)}{4} d\theta^2, \quad r \in [0, \frac{\pi}{2}],$$
with coordinates
$$(r, e^{i\theta}) \mapsto \left(\frac{1}{2}\cos(2r), \frac{1}{2}\sin(2r)e^{i\theta} \right).$$
- The Hopf fibration in these coordinates, therefore, looks like
$$(t, e^{i\theta_1}, e^{i\theta_2}) \mapsto (t, e^{i(\theta_1-\theta_2)}).$$

- Now on $\mathbb{S}^3(1)$ we have an orthognal frame

$$\left\{\partial_{\theta_1}+\partial_{\theta_2},\ \partial_t,\ \frac{\cos^2(t)\partial_{\theta_1}-\sin^2(t)\partial_{\theta_2}}{\cos(t)\sin(t)}\right\},$$

where the first vector is tangent to the Hopf fiber and the two other vectors have unit length.
- On $\mathbb{S}^2(\frac{1}{2})$

$$\left\{\partial_r, \frac{2}{\sin(2r)}\partial_\theta\right\}$$

is an orthonormal frame.
- The Hopf map clearly maps

$$\partial_t \mapsto \partial_r,$$
$$\frac{\cos^2(t)\partial_{\theta_1}-\sin^2(t)\partial_{\theta_2}}{\cos(t)\sin(t)} \mapsto \frac{\cos^2(r)\partial_\theta+\sin^2(r)\partial_\theta}{\cos(r)\sin(r)} = \frac{2}{\sin(2r)}\cdot\partial_\theta,$$

thus showing that it is an isometry on vectors perpendicular to the fiber.

- Note that the map

$$(t, e^{i\theta_1}, e^{i\theta_2}) \mapsto (t, e^{i(\theta_1-\theta_2)}) \mapsto \begin{pmatrix} \cos(t)e^{i\theta_1} & -\sin(t)e^{i\theta_2} \\ \sin(t)e^{-i\theta_2} & \cos(t)e^{-i\theta_1} \end{pmatrix}$$

gives us the isometry from $\mathbb{S}^3(1)$ to $\mathrm{SU}(2)$.

- The map $(t, e^{i\theta_1}, e^{i\theta_2}) \mapsto (t, e^{i(\theta_1-\theta_2)})$ from $I\times\mathbb{S}^1\times\mathbb{S}^1$ to $I\times\mathbb{S}^1$ is actually always a Riemannian submersion when the domain is endowed with the doubly warped product metric

$$dt^2+\varphi^2(t)d\theta_1^2+\psi^2(t)d\theta_2^2$$

and the target has the rotationally symmetric metric

$$dr^2+\frac{(\varphi(t)\cdot\psi(t))^2}{\varphi^2(t)+\psi^2(t)}d\theta^2.$$

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- This submersion can be generalized to higher dimensions as follows.

⊙ On $I\times\mathbb{S}^{2n+2}\times\mathbb{S}^1$ consider the doubly warped product metric

$$dt^2+\varphi^2(t)ds^2_{2n+1}+\psi^2(t)d\theta^2.$$

The unit circle acts by complex scalar multiplication on both $\mathbb{S}^{2n+1}$ and $\mathbb{S}^1$, and consequently induces a free isometric action on the space: if $\lambda\in\mathbb{S}^1$ and $(z,w)\in\mathbb{S}^{2n+1}\times\mathbb{S}^1$, then $\lambda\cdot(z,w)=(\lambda z,\lambda w)$.

⊙ The quotient map

$$I\times\mathbb{S}^{2n+1}\times\mathbb{S}^1\to I\times((\mathbb{S}^{2n+1}\times\mathbb{S}^1)/\mathbb{S}^1)$$

can be made into a Riemannian submersion by choosing suitable metric on the quotient space.

— To find the metric, we split the canonical metric

$$ds^2_{2n+1} = h + g,$$

where h corresponds to the metric along the Hopf fiber and g the orthogonal complement.

– In other words, if $\widehat{\pi} : T_p\mathbb{S}^{2n+1} \to (T_p\mathbb{S}^{2n+1})^V$ is the orthogonal projection (with respect to ds^2_{2n+1}) whose image is the distribution generated by the Hopf action, then

$$h(v, w) = ds^2_{2n+1}(\widehat{\pi}_* v, \widehat{\pi}_* w)$$

and

$$g(v, w) = ds^2_{2n+1}(v - \widehat{\pi}_* v, w - \widehat{\pi}_* w).$$

— We can then define

$$dt^2 + \varphi^2(t) ds^2_{2n+1} + \psi^2(t) d\theta^2 = dt^2 + \varphi^2(t) g + \varphi^2(t) h + \psi^2(t) d\theta^2.$$

Now notice that

$$(\mathbb{S}^{2n+1} \times \mathbb{S}^1)/\mathbb{S}^1 = \mathbb{S}^{2n+1}$$

and that $\mathbb{S}^1$ only collapses the Hopf fiber while leaving the orthogonal component to the Hopf fiber unchanged.

In analogy with the above example, we therefore obtain that the metric on $I \times \mathbb{S}^{2n+1}$ can be written

$$ds^2 + \varphi^2(t) g + \frac{(\varphi(t) \cdot \psi(t))^2}{\varphi^2(t) + \psi^2(t)} h.$$

(i) In the case when $n = 0$, we recapture the previous case, as g does not appear.

(ii) When $n = 1$, the decomposition $ds^2_3 = h + g$ can also be written

$$ds^2_3 = (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2,$$

where $(\sigma^1)^2 = h$, $(\sigma^2)^2 + (\sigma^3)^2 = g$, and $\{\sigma^1, \sigma^2, \sigma^3\}$ is the coframing coming from the identification $\mathbb{S}^3 \cong SU(2)$.

– The Riemannian submersion in this case can therefore be written

$$(I \times \mathbb{S}^3 \times \mathbb{S}^1, dt^2 + \varphi^2(t)[(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2] + \psi^2(t) d\theta^2)$$

$$\downarrow$$

$$(I \times \mathbb{S}^3 \times \mathbb{S}^1, dt^2 + \varphi^2(t)[(\sigma^2)^2 + (\sigma^3)^2] + \tfrac{(\varphi(t)\cdot\psi(t))^2}{\varphi^2(t)+\psi^2(t)} (\sigma^1)^2).$$

(iii) If we let $\varphi = \sin(t)$, $\psi = \cos(t)$ and $t \in I = [0, \pi/2]$, then we obtain the generalized Hopf fibration

$$\mathbb{S}^{2n+3} \to \mathbb{CP}^{n+1}$$

defined by

$$(0, \frac{\pi}{2}) \times (\mathbb{S}^{2n+1} \times \mathbb{S}^1) \to (0, \frac{\pi}{2}) \times (\mathbb{S}^{2n+1} \times \mathbb{S}^1/\mathbb{S}^1)$$

as a Riemannian submersion, and the Fubini-Study metric on $\mathbb{CP}^{n+1}$ can be represented as

$$dt^2 + \sin^2(t)(g + \cos^2(t) h).$$