Integral (or Weak) Solutions

1. The Method of Characteristics Revisited

• The method of characteristics applied to the problem

$$\begin{cases} u_t + q(u)_x = 0 \\ u(x,0) = g(x) \end{cases} \tag{1}$$

gives the travelling wave

$$u(x,t) = g(x - q'(g(\xi)t)), \qquad \left(q' = \frac{dq}{du}\right) \tag{2}$$

with local speed $q'(g(\xi))$, in the positive x-direction.

⊙ Since $u(x,t) \equiv g(\xi)$ along the characteristic based at $(\xi, 0)$, from (2) we obtain that u is implicitly defined by the equation

$$G(x,t,u) \equiv u - g(x - q'(u)t) = 0. \tag{3}$$

If g and q' are smooth, the **implicit function theorem** implies that the equation (3) defines u as a function of (x,t), as long as the condition

$$G_u(x,t,u) = 1 + tq''(u)g'(x - q'(u)t) \neq 0 \tag{4}$$

holds. As immediate consequence is that if q'' and g' have the same sign, the solution given by the method of characteristics is defined and smooth for all times $t \geq 0$. Precisely, we have

Proposition 1. *Suppose that $q \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$ and $g'q'' \geq 0$ in $\mathbb{R}$. Then (3) defines the unique solution of problem (1) in the half-plane $t \geq 0$.*
Moreover, $u(x,t) \in C^1(\mathbb{R} \times [0,\infty))$.

• Thus, if q'' and g' have the same sign, the characteristics do not intersect.

• Note that in the ε-approximation of the green light problem, q is concave and g_ε is decreasing.

– Although g_ε is not smooth, the characteristics do not intersect and ρ_ε is well defined for all times $t > 0$.
In the limit as $\varepsilon \to 0$, the discontinuity of g reappears and the fan of characteristics produces a rarefaction waves.

• What happens if q'' and g' have a different sign in an interval $[a,b]$?

⊙ Proposition 1 still holds for all small times, since $G_u \sim 1$ if $t \sim 0$, but when the time goes on we expect the formation of a shock.

– Indeed, suppose, for instance, that q is concave and g is increasing.
The family of characteristics based on a point in the interval $[a,b]$ is

$$x = q'(g(\xi))t + \xi \quad \xi \in [a,b]. \tag{5}$$

When ξ increases, g increases as well, while $q'(g(\xi))$ decrease so that we expect intersection of characteristics along a shock curve.

– The main question is to find the positive time t_s (**breaking time**) and the location x_s of the **first appearance of the shock**.

- According to the above discussion, the **breaking time** must coincide with the first time t at which the expression
$$G_u(x,t,u) = 1 + tq''(u)g'(x - q'(u)t)$$
becomes zero.

– Computing G_u along the characteristic (5), we have $u = g(\xi)$ and
$$G_u(x,t,u) = 1 + tq''(u)g'(\xi).$$
Assume that the nonnegative function
$$z(\xi) = -q''(g(\xi))g'(\xi)$$
attains its maximum only at the point $\xi_M \in [a,b]$. Then $z(\xi_M) > 0$ and
$$t_s = \min_{\xi\in[a,b]} \frac{1}{z(\xi)} = \frac{1}{z(\xi_M)}. \tag{6}$$
Since x_s belongs to the characteristics $x = q'(g(\xi_M)) + \xi_M$, we find
$$x_s = \frac{q'(g(\xi_M))}{z(\xi_M)} + \xi_M. \tag{7}$$

- The point (x_s, t_s) has an interesting **geometric meaning**:
If $q''g' < 0$, the family of characteristics (5) admits an **envelope** and (x_s, t_s) is the point on the envelope with minimum time coordinate, recalling the following

Definition. *The* **envelope** *of a family of curves* $\phi(x,t,\xi) = 0$*, with parameter* ξ*, is a curve* $\psi(x,t) = 0$ **tangent at each one of its points to a curve of the family**. *If the family of curves* $\phi(x,t,\xi) = 0$ *has an envelope, its parametric equations are obtained by solving the system* $\begin{cases} \phi(x,t,\xi) = 0 \\ \phi_\xi(x,t,\xi) = 0 \end{cases}$ *with respect to* x, t.

Example 1. Consider the initial value problem
$$\begin{cases} u_t + (1-2u)u_x = 0, \\ u(x,0) = \arctan x. \end{cases}$$
We have $q(u) = u - u^2$, $g(\xi) = \arctan\xi$, and hence $q'(u) = 1 - 2u$, $q''(u) = -2$ and $g'(\xi) = \frac{1}{1+\xi^2}$. Therefore, the function
$$z(\xi) = -q''(g(\xi))g'(\xi) = \frac{2}{1+\xi^2}$$
has a maximum at $\xi_M = 0$ and $z(0) = 2$.

⊙ The breaking time is $t = \frac{1}{2}$ and $x_s = \frac{1}{2}$. Thus, the shock curve starts from $(\frac{1}{2}, \frac{1}{2})$.

– For $0 \le t < \frac{1}{2}$, the solution u is smooth and implicitly defined by the equation
$$u - \arctan[x - (1-2u)t] = 0. \tag{8}$$
After $t = \frac{1}{2}$, equation (8) defines u as a multivalued function of (x,t) and does not define a solution anymore.

⊙ How does the solution evolve after $t = 1/2$?

– We have to insert a shock wave into the multivalued graph of u in such a way the conservation law is preserved.

– We will see that the correct insertion point is prescribed by the Rankine-Hugoniot condition.

2. Definition of Integral Solutions.

- We have seen that the method of characteristics is **not** sufficient, in general, to determine the solution of an initial value problem for all times $t > 0$.
- – In the green light problem a rarefaction wave was used to construct the solution in a region not covered by characteristics.
- – In the traffic jam case the solution undergoes a shock, propagating according to the Rankine-Hugoniot condition.

- Some questions arise naturally:

  Q1. *In which sense is the differential equation satisfied across a shock or, more generally, across a separation curve where the constructed curve is not differentiable?*

  Q2. *In the solution unique?*

  Q3. *If there is no uniqueness, is there a criterion to select the "physically correct"* **solution?**

- To answer, we need first of all to introduce a more flexible notion of solutions, in which the derivatives of the solution are not directly involved.
- – Let us go back to the problem

$$\begin{cases} u_t + q(u)_x = 0 \\ u(x,0) = g(x) \end{cases} \tag{9}$$

and assume for the moment that u is a smooth solution, at least of class C^1 in $\mathbb{R}^n \times [0,\infty)$. We say that u is a **classical solution**.

- Let v be a smooth function in $\mathbb{R} \times [0,+\infty)$ with compact support.
- – Multiply the differential equation by u and integrate on $\mathbb{R} \times (0,+\infty)$. We obtain

$$\int_0^\infty \int_{\mathbb{R}} [u_t + q(u)_x] v \, dxdt = 0. \tag{10}$$

The idea is to carry the derivatives onto the test function v via an integration by parts.

- – If we integrate by parts the first term in (10) with respect to t, we obtain

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} u_t v \, dxdt &= -\int_0^\infty \int_{\mathbb{R}} u v_t \, dxdt - \int_{\mathbb{R}} u(x,0)v(x,0)\, dx \\ &= -\int_0^\infty \int_{\mathbb{R}} u v_t \, dxdt - \int_{\mathbb{R}} g(x)v(x,0)\, dx. \end{aligned}$$

- – If we integrate by parts the second term in (10) with respect to x, we obtain

$$\int_0^\infty \int_{\mathbb{R}} q(u)_x v \, dxdt = -\int_0^\infty \int_{\mathbb{R}} q(u) v_x \, dxdt$$

- – Then the equation (10) becomes

$$\int_0^\infty \int_{\mathbb{R}} [u v_t + q(u) v_x] \, dxdt + \int_{\mathbb{R}} g(x)v(x,0) = 0. \tag{11}$$

We have obtained an integral equation, valid **for every test function** v.

- – Observe that **no derivative of** u appears in (11).

- On the other hand, suppose that a smooth function u satisfies (11) **for every test function** v.
- Integrating by parts in the reverse order, we arrive at the equation

$$\int_0^\infty \int_{\mathbb{R}} [u_t + q(u)_x] v \, dxdt + \int_{\mathbb{R}} [g(x) - u(x,0)] v(x,0) = 0, \tag{12}$$

still true for every test function v.
- Choose v vanishing for $t = 0$; then the second integral is zero and the arbitrariness of v implies

$$u_t + q(u)_x = 0 \quad \text{in } \mathbb{R} \times (0, +\infty). \tag{13}$$

- Choosing now $v(x,0) \neq 0$, from (12) and (13), we obtain

$$\int_{\mathbb{R}} [g(x) - u(x,0)] v(x,0) = 0.$$

The arbitrariness of v implies

$$u(x,0) = g(x) \quad \text{in } \mathbb{R}.$$

Therefore u is a solution of problem (9).

Conclusion. *A function $u \in C^1(\mathbb{R} \times [0, +\infty))$ is a solution of problem (9) iff the equation (11) holds for every function v.*

- But (11) makes sense for u merely bounded, so that it constitutes an alternative **integral** or **weak** formulation of problem (9).
- This motivates the following definition.

Definition 1. *A function u, bounded in $\mathbb{R} \times [0, \infty)$, is called integral (or weak) solution of problem (9) if equation (11) holds for every test function v in $\mathbb{R} \times [0, \infty)$, with compact support.*

- We point out that an integral solution may be discontinuous, since the definition requires only boundedness.

3. The Rankine-Hugoniot Condition.

• Definition 1 looks rather satisfactory, because of its flexibility.
– However we have to understand which information about the behavior of weak solutions at a singularity, e.g. across a shock curve, is hidden in the integral formulation.

• Consider an open set V, contained in the half-plane $t > 0$, partitioned into two disjoint domains V^+ and V^- by a smooth (shock) curve Γ of equation $x = s(t)$.

• Suppose u is a weak solution in V, of class C^1 in both $\overline{V^+}$ and $\overline{V^-}$, separately.
– We have seen that u is a classical solution of

$$u_t + q(u)_x = 0 \quad \text{in } V^+ \text{ and } V^-.$$

⊙ Choose now a test function v, supported in a compact set $K \subset V$, such that $K \cap \Gamma$ is non-empty.
– Since $v(x,0) = 0$, we can write

$$\begin{aligned} 0 &= \int_0^\infty \int_{\mathbb{R}} [uv_t + q(u)v_x]\, dxdt \\ &= \int_{V^+} [uv_t + q(u)v_x]\, dxdt + \int_{V^-} [uv_t + q(u)v_x]\, dxdt. \end{aligned}$$

Integrating by parts and observing that $v = 0$ on $\partial V^+ \setminus \Gamma$, we have

$$\begin{aligned} \int_{V^+} [uv_t + q(u)v_x]\, dxdt &= -\int_{V^+} [u_t + q(u)_x]v\, dxdt + \int_\Gamma [u_+ n_2 + q(u_+)n_1]v\, d\ell \\ &= \int_\Gamma [u_+ n_2 + q(u_+)n_1]v\, d\ell \end{aligned}$$

where u_+ denotes the value of u on Γ from the V^+ side, $\mathbf{n} = (n_1, n_2)$ is the outward unit normal vector on ∂V^+ and $d\ell$ denotes the arc length on Γ. Similarly, since $\mathbf{n}$ is inward with respect to V^-:

$$\int_{V^-} [uv_t + q(u)v_x]\, dxdt = -\int_\Gamma [u_- n_2 + q(u_-)n_1]v\, d\ell$$

where u_- denotes the value of u on Γ from the V^- side. Therefore we deduce that

$$\int_\Gamma \{[q(u_+) - q(u_-)]n_1 + [u_+ - u_-]n_2\}v\, d\ell = 0.$$

The arbitrariness of v yields

$$[q(u_+) - q(u_-)]n_1 + [u_+ - u_-]n_2 = 0 \quad \text{on } \Gamma. \tag{14}$$

If u is continuous across Γ, (14) is automatically satisfied.

If $u_+ \neq u_-$, we write the relation (14) more explicitly. Namely, since $x = s(t)$ on Γ, we have

$$\mathbf{n} = (n_1, n_2) = \frac{1}{\sqrt{1 + (\dot{s}(t))^2}}(-1, \dot{s}(t)).$$

Hence (14) becomes, after simple calculations,

$$\dot{s} = \frac{q[u_+(s,t)] - q[u_-(s,t)]}{u_+(s,t) - u_-(s,t)}, \tag{15}$$

which is the **Rankine-Hugoniot** for the shock curve Γ.

- In general, functions constructed by connecting solutions and rarefaction waves in a continuous way are weak solutions.
- The same is true for shock waves satisfying the Rankine-Hugoniot condition.
- Then, the solutions of the green light and of the traffic jam problems are precisely integral solutions.

- Thus, Definition 1 gives a satisfactory answer to question **Q1**.
  The other two questions require a deeper analysis as the following example shows.

Example 2 (Non-uniqueness). Imagine a flux of particles along the x-axis, each one moving with constant speed.
Suppose that $u = u(x,t)$ represents the *velocity field*, which gives the speed of the particle located at x at time t.
If $x = x(t)$ is the *path* of a particle, its velocity at time t is given by

$$\dot{x}(t) = u(x(t), t) \equiv \text{constant}.$$

Thus, we have

$$\begin{aligned} 0 =& \frac{d}{dt}u(x(t),t) = u_t(x(t),t) + u_x(x(t),t)\dot{x}(t) \\ =& u_t(x(t),t) + u_x(x(t),t)u(x(t),t). \end{aligned}$$

Therefore $u = u(x,t)$ satisfies **Burger's equation**

$$u_t + uu_x = u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{16}$$

which is a conservation law with $q(u) = \frac{u^2}{2}$.
- Note that q is strictly convex: $q'(u) = u$ and $q''(u) = 1$.
  We couple (16) with the initial condition $u(x,0) = g(x)$, where

$$g(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases}$$

The characteristics are the straight lines

$$x = g(x_0)t + x_0. \tag{17}$$

Therefore,

$$u(x,t) = \begin{cases} 0 & x < 0 \\ 1 & x > t. \end{cases}$$

The region $S = \{0 < x < t\}$ is not covered by characteristics.

– As in the green light problem, we connect the states 0 and 1 through a **rarefaction wave**.

– Since $q'(u) = u$, we have $r(s) = (q')^{-1}(s) = s$, so that we construct the weak solution

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t}, & 0 < x < t \\ 1 & x > t. \end{cases} \tag{18}$$

However, u is **not the unique weak solution**.

- There exists also a **shock wave** solution.
  In fact, since

$$u_- = 0, \quad u_+ = 1, \quad q(u_-) = 0, \quad q(u_+) = \frac{1}{2},$$

the Rankine-Hugoniot condition yields

$$\dot{s}(t) = \frac{q(u_+) - q(u_-)}{u_+ - u_-} = \frac{1}{2}.$$

Given the discontinuity at $x = 0$ of the initial data, the shock curve starts at $s(0) = 0$ and it is the straight line

$$x = \frac{t}{2}.$$

Hence the function

$$u(x,t) = \begin{cases} 0 & x < \frac{t}{2} \\ 1 & x > \frac{t}{2}. \end{cases}$$

is another weak solution.

– As we will see, this shock wave has to be considered not physically acceptable.

- The example shows that the answer to question **Q2** is negative and the question **Q3** becomes relevant.

– We need a criterion to establish which one is the physically correct solution.

4. The Entropy Condition

- From Proposition 2 we have seen that the equation

$$G(x,t,u) \equiv u - g(x - q'(u)t) = 0$$

defines the unique classical solution u of problem (9), at least for small times. The **implicit function theorem** gives

$$u_x = -\frac{G_x}{G_u} = \frac{g'}{1 + tg'q''}.$$

If we assume $g' > 0$, $q'' \geq C > 0$. we obtain

$$u_x \leq \frac{E}{t}, \quad \text{where } E = \frac{1}{C}.$$

Using the mean value theorem, we deduce the following condition, called

Entropy Condition. *There exists $E \geq 0$ such that, for every x, $z \in \mathbb{R}$, $z > 0$, and every $t > 0$,*

$$u(x+z,t) - u(x,t) \leq \frac{E}{t} z. \tag{19}$$

- The denomination comes from an analogy with gas dynamics, where a condition like (19) implies that entropy increases across a shock.
- The entropy condition does not involve any derivative of u and makes perfect sense for discontinuous solutions as well.
- The function

$$x \mapsto u(x,t) - \frac{E}{t} x$$

is **decreasing**. In fact, let $x + z = x_2$, $x = x_1$ and $z > 0$. Then $x_2 > x_1$ and (19) is equivalent to

$$u(x_2,t) - \frac{E}{t} x_2 \leq u(x_1,t) - \frac{E}{t} x_1. \tag{20}$$

- If x is a discontinuity point for $u(\cdot, t)$, then

$$u_+(x,t) < u_-(x,t) \tag{21}$$

where $u_\pm(x,t) = \lim_{y \to x^\pm} u(y,t)$. In fact, we may choose $x_1 < x < x_2$ and let x_1 and x_2 both go to x in (20) to obtain (21).

- If q is **strictly convex**, (21) yields

$$q'(u_+) < \frac{q(u_+) - q(u_-)}{u_+ - u_-} < q'(u_-).$$

Then, the Rankine-Hugoniot solution implies that, if $x = s(t)$ is a shock wave

$$q'(u_+(x,t)) < \dot{s}(t) < q'(u_-(x,t)) \tag{22}$$

which is called the **entropy inequality**. The geometrical meaning of (21) is remarkable:

The slope of a shock curve is less than the slope of the left-characteristics and greater than the slope of the right-characteristics.

Roughly speaking, the characteristics **hit forward** in time the shock line, so that it is not possible to go back in time along characteristics and hit a shock line, expressing a sort of irreversibility after a shock.

- Thus, in Example 2, the solution w represents a non-physical shock since it does not satisfy the entropy condition.
  The correct solution is therefore the simple wave (18).
- The following important result holds.

Theorem 2. *If $q \in C^2(\mathbb{R})$ is convex or concave and g is bounded, there exists a unique entropy solution of the problem (9).*

5. The Riemann Problem.

- We apply Theorem 2 to solve explicitly the problem (9) with initial data

$$g(x) = \begin{cases} u_+ & x < 0 \\ u_- & x > 0, \end{cases} \tag{23}$$

  where u_+ and u_- are constants, $u_+ \neq u_-$.
  This problem is known as **Riemann problem**, and it is particularly important for the numerical approximation of more complex problems.

Theorem 3. *Let $q \in C^2(\mathbb{R})$ be strictly convex and $q'' \geq h > 0$.*
(a) If $u_+ < u_-$, the unique entropy solution is the shock wave

$$u(x,t) = \begin{cases} u_+ & \frac{x}{t} > \dot{s}(t) \\ u_- & \frac{x}{t} < \dot{s}(t) \end{cases} \tag{24}$$

where

$$\dot{s}(t) = \frac{q(u_+) - q(u_-)}{u_+ - u_-}.$$

(a) If $u_+ > u_-$, the unique entropy solution is the rarefaction wave

$$u(x,t) = \begin{cases} u_- & \frac{x}{t} < q'(u_-) \\ r(\frac{x}{t}) & q'(u_-) < \frac{x}{t} < q'(u_+) \\ u_+ & \frac{x}{t} > q'(u_+) \end{cases}$$

where $r = (q')^{-1}$ is the inverse function of q'.

Proof. **(a)** The shock wave (24) satisfies the Rankine-Hugoniot condition and therefore it is clearly a weak solution.
– Moreover, since $u_+ < u_-$, the entropy condition holds as well, and u is the unique entropy solution of problem (23) by Theorem 2.
(b) Since

$$r(q'(u_+)) = u_+ \quad \text{and} \quad r(q'(u_-)) = u_-,$$

u is continuous in the half-plane $t > 0$ and we have only to check that u satisfies the equation $u_t + q(u)_x = 0$ in the region

$$S = \{(x,t) : q'(u_-) < \frac{x}{t} < q'(u_+)\}.$$

Let $u(x,t) = r(\frac{x}{t})$. We have

$$u_t + q(u)_x = -r'\left(\frac{x}{t}\right)\frac{x}{t^2} + q'(r)r'\left(\frac{x}{t}\right)\frac{1}{t} = r'\left(\frac{x}{t}\right)\frac{1}{t}\left[q'(r) - \frac{x}{t}\right] = 0.$$

Thus u is a weak solution in the upper half-space.

- **Check the entropy condition.**

Case 1. $q'(u_-)t < x < x+z < q'(u_+)t$.
Since $q'' \geq h > 0$, we have

$$r'(s) = \frac{1}{q''(r)} \leq \frac{1}{h}, \qquad \text{where } s = q'(r),$$

so that

$$\begin{aligned} u(x+z,t) - u(x,t) =& r\left(\frac{x+z}{t}\right) - r\left(\frac{x}{t}\right) \\ =& r'\left(\frac{x+z^*}{t}\right)\frac{z}{t} \\ \leq& \frac{1}{h}\frac{z}{t}, \qquad (0 < z^* < z), \end{aligned}$$

which is the entropy condition with $E = 1/h$.

Case 2. $x < x+z < q'(u_-)t$. We have $u(x+z) = u(x)$.

Case 3. $q'(u_+)t < x < x+z$. We have $u(x+z) = u(x)$.

Case 4. $x < q'(u_-)t < q'(u_+)t < x+z$. We have

$$u(x+z) = u_+ < r(\frac{x+z}{t}) \quad \text{and} \quad u(x) = u_- > r(\frac{x}{t}).$$

Case 5. $x < q'(u_-)t < x+z < q'(u_+)t$. We have

$$u(x+z) = r(\frac{x+z}{t}) \quad \text{and} \quad u(x) = u_- > r(\frac{x}{t}).$$

Case 6. $q'(u_-)t < x < q'(u_+)t < x+z$. We have

$$u(x+z) = u_+ < r(\frac{x+z}{t}) \quad \text{and} \quad u(x) = \frac{x}{t}. \qquad \square$$