The Multi-dimensional Wave Equation ($n > 1$)

Special Solutions: Spherical waves ($n = 3$)

Spherical waves ($n = 3$) are of the form

$$u(\mathbf{x}, t) = w(r, t)$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

– In this case, it is appropriate to use the spherical coordinates

$$x_1 = r\cos\theta\sin\psi, \quad x_2 = r\sin\theta\sin\psi, \quad x_3 = r\cos\psi.$$

Then, the wave equation becomes

$$\frac{1}{c^2}u_{tt} - u_{rr} - \frac{2}{r}u_r - \frac{1}{r^2}\left\{\frac{1}{(\sin\psi)^2}u_{\theta\theta} + u_{\psi\psi} + \frac{\cos\psi}{\sin\psi}u_\psi\right\} = 0. \tag{1}$$

- Let us now determine the general form of a spherical wave in $\mathbb{R}^3$.
  Inserting $u(\mathbf{x}, t) = w(r, t)$ into (1), we obtain

$$w_{tt} - c^2\left(w_{rr} + \frac{2}{r}w_r\right) = 0$$

which can be written in the form

$$(rw)_{tt} - c^2(rw)_{rr} = 0. \tag{2}$$

Then, d'Alembert's formula gives

$$w(r, t) = \frac{F(r + ct)}{r} + \frac{G(r - ct)}{r} \equiv w_i(r, t) + w_o(r, t) \tag{3}$$

which represents the superposition of two attenuated progressive spherical waves.

– The wave fronts of w_o are the spheres $r - ct = k$, expanding as time goes on. Hence, w_o represents an **outgoing wave**.

– The wave fronts of w_i are the spheres $r + ct = k$, contracting as time goes on. Hence, w_i represents an **incoming wave**.

The Cauchy Problem

1. Fundamental Solution ($n = 3$) and Strong Huygens' Principle.

- In this section we consider the global Cauchy problem for the three-dimensional homogeneous wave equation:

$$\begin{cases} u_{tt} - c^2\Delta u = 0, & x \in \mathbb{R}^3, \quad t > 0, \\ u(x, 0) = g(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = h(\mathbf{x}), & x \in \mathbb{R}^3. \end{cases} \tag{4}$$

– We know that problem (4) has at most one solution $u \in C^2(\mathbb{R}^3 \times [0, +\infty))$.

– Our purpose here is to show that the solution u exists and to find an explicit formula for it, in terms of g and h.

– Our derivation is rather heuristic so that, for the time being, we do not worry too much about the correct hypotheses on h and g, which we assume as smooth as we need to carry out the calculations.

- First we need a lemma that reduces the problem to the case $g = 0$.
  Denote by w_h the solution of the problem

$$\begin{cases} w_{tt} - c^2\Delta w = 0, & x \in \mathbb{R}^3, \quad t > 0, \\ w(\mathbf{x}, 0) = 0, \quad w_t(\mathbf{x}, 0) = h(\mathbf{x}), & x \in \mathbb{R}^3. \end{cases} \tag{5}$$

Lemma 1. *Denote by* w_h *the solution of the problem (5). Let* $w_g \in C^3$*, then* $v = \partial_t w_g$ *solves the problem*

$$\begin{cases} w_{tt} - c^2 \Delta w = 0, & x \in \mathbb{R}^3, \quad t > 0, \\ w(x,0) = g, \quad w_t(\mathbf{x},0) = 0, & x \in \mathbb{R}^3. \end{cases}$$

Therefore the solution to the problem

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, & x \in \mathbb{R}^3, \quad t > 0, \\ u(x,0) = g, \quad u_t(0,t) = h, & x \in \mathbb{R}^3. \end{cases}$$

is given by

$$u(x,t) = \frac{\partial}{\partial t} w_g(x,t) + w_h(x,t). \tag{6}$$

Proof. Let $v = \partial_t w_g$. Differentiating the wave equation with respect to t we have

$$0 = \partial_t(\partial_{tt} w_g - c^2 \Delta w_g) = (\partial_{tt} w_g - c^2 \Delta)\partial_t w_g = v_{tt} - c^2 \Delta v.$$

Moreover,

$$v(x,0) = \partial_t w_g(x,0) = g(x), \quad v_t(x,0) = \partial_{tt} w_g(x,0) = c^2 \Delta w_g(x,0) = 0. \qquad \square$$

- The lemma shows that, once the solution of (5) is determined, the solution of the complete problem (4) is given by (6).

- Therefore, we focus on the solution of (2), first with a special h, given by the three-dimensional Dirac measure at $\mathbf{y}$, $\delta(\mathbf{x} - \mathbf{y})$.

⊙ For example, in the case of sound waves, this initial data models a sudden change of the air density, concentrated at a point $\mathbf{y}$.

– If w represents the density variation with respect to a static atmosphere, then w solves the problem

$$\begin{cases} w_{tt} - c^2 \Delta w = 0, & x \in \mathbb{R}^3, \quad t > 0, \\ w(\mathbf{x},0) = 0, \quad w_t(\mathbf{x},0) = \delta(\mathbf{x} - \mathbf{y}), & x \in \mathbb{R}^3. \end{cases} \tag{7}$$

- The solution of (7), which we denote by $K(\mathbf{x}, \mathbf{y}, t)$, is called **fundamental solution** of the three-dimensional wave equation.

– To solve (7), we use the heat equation, approximating the Dirac measure with the *fundamental solution* of the three-dimensional diffusion equation.

– Indeed, we know that

$$\Gamma(\mathbf{x} - \mathbf{y}, \varepsilon) = \frac{1}{(4\varepsilon\pi)^{\frac{3}{2}}} \exp\left(-\frac{|\mathbf{x} - \mathbf{y}|^2}{4\varepsilon}\right) \to \delta(\mathbf{x} - \mathbf{y}), \quad \text{as } \varepsilon \to 0.$$

– Denote by w_ε the solution of

$$\begin{cases} w_{tt} - c^2 \Delta w = 0, & x \in \mathbb{R}^3, \quad t > 0, \\ w(\mathbf{x}, 0) = 0, \quad w_t(\mathbf{x}, 0) = \Gamma(\mathbf{x} - \mathbf{y}, \varepsilon), & x \in \mathbb{R}^3. \end{cases}$$

– Since $\Gamma(\mathbf{x} - \mathbf{y}, \varepsilon)$ is radially symmetric with pole at $\mathbf{y}$, we expect that w_ε shares the same type of symmetry and is a spherical wave of the form $w_\varepsilon = w_\varepsilon(r, t)$, $r = |\mathbf{x} - \mathbf{y}|$. Thus, we write

$$w_\varepsilon(r, t) = \frac{F(r + ct)}{r} + \frac{G(r - ct)}{r}. \tag{8}$$

The initial conditions require

$$F(r) + G(r) = 0 \quad \text{and} \quad c(F'(r) - G'(r)) = r\Gamma(r, \varepsilon)$$

i.e.

$$F = -G \quad \text{and} \quad G'(r) = -\frac{r\Gamma(r, \varepsilon)}{2c}$$

Integrating the second relation yields

$$\begin{aligned} G(r) &= -\frac{1}{2c(4\pi\varepsilon)^{3/2}} \int_0^r s \exp\left(-\frac{s^2}{4\varepsilon}\right) ds \\ &= \frac{1}{4\pi c} \frac{1}{\sqrt{4\pi\varepsilon}} \left[\exp\left(-\frac{r^2}{4\varepsilon}\right) - 1\right] \end{aligned}$$

and finally

$$w_\varepsilon(r, t) = \frac{1}{4\pi c r} \left[\frac{1}{\sqrt{4\pi\varepsilon}} \exp\left(-\frac{(r - ct)^2}{4\varepsilon}\right) - \frac{1}{\sqrt{4\pi\varepsilon}} \exp\left(-\frac{(r + ct)^2}{4\varepsilon}\right)\right]$$

– Now observe that the function

$$\widetilde{\Gamma}(r, \varepsilon) = \frac{1}{\sqrt{4\pi\varepsilon}} \exp\left(-\frac{r^2}{4\varepsilon}\right)$$

is the fundamental solution of the **one-dimensional** heat equation with $x = r$ and $t = \varepsilon$. Letting $\varepsilon \to 0$ we find

$$w_\varepsilon(r, t) \to \frac{1}{4\pi c r}[\delta(r - ct) - \delta(r + ct)].$$

Since $r + ct > 0$ for every $t > 0$, we deduce that $\delta(r + ct) = 0$ and therefore we conclude that

$$K(\mathbf{x}, \mathbf{y}, t) = \frac{\delta(r - ct)}{4\pi c r}, \quad r = |\mathbf{x} - \mathbf{y}|. \tag{9}$$

Thus, the fundamental solution is an **outgoing travelling wave**, initially concentrated at $\mathbf{y}$ and thereafter on

$$\partial B_{ct}(\mathbf{y}) = \{\mathbf{x} : |\mathbf{x} - \mathbf{y}| = ct\}.$$

The union of the surfaces $\partial B_{ct}(\mathbf{y})$ is called the **support** of K and coincides with the **boundary** of the *forward space-time cone, with vertex at* $(\mathbf{y}, 0)$ and opening $\theta = \tan^{-1} c$, given by

$$C^*_{\mathbf{y},0} = \{(\mathbf{x}, t) : |\mathbf{x} - \mathbf{y}| \le ct, \quad t > 0\}.$$

In other words, $\partial C^*_{\mathbf{y},0}$ constitutes the **range of influence of the point $\mathbf{y}$**.

- The fact that the range of influence of the point $\mathbf{y}$ is only the **boundary** of the forward cone and **not the full** cone has important consequences on the nature of the disturbances governed by the three-dimensional wave equation.
- The most striking phenomenon is that a perturbation generated at time $t = 0$ by a point source placed at $\mathbf{y}$ is felt at the point $\mathbf{x}_0$ **only at the time** $t_0 = \frac{|\mathbf{x}_0 - \mathbf{y}|}{c}$.

– This is known as **strong Huygens' principle** and explains why **sharp signals** are propagates from a point source.

- We will shortly see that this is not the case in two dimensions.

The Kirchhoff formula

- Using the fundamental solution, we may deduce a formula for the solution of (2) with a general h. We may write

$$h(\mathbf{x}) = \int_{-\infty}^{+\infty} \delta(\mathbf{x} - \mathbf{y}) h(\mathbf{y}) d\mathbf{y},$$

looking at $h(\mathbf{x})$ as a superposition of impulses $\delta(\mathbf{x} - \mathbf{y})h(\mathbf{y})$ located at $\mathbf{y}$, of strength $h(\mathbf{y})$. Accordingly, the solution of (2) is given by the superposition of the corresponding solution $K(\mathbf{x}, \mathbf{y}, y)h(\mathbf{y})$, that is,

$$\begin{aligned} w_h(\mathbf{x}, t) &= \int_{\mathbb{R}^3} K(\mathbf{x}, \mathbf{y}, t) h(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^3} \frac{\delta(|\mathbf{x} - \mathbf{y}| - ct)}{4\pi c |\mathbf{x} - \mathbf{y}|} h(\mathbf{y}) d\mathbf{y} \\ &= \int_0^{\infty} \frac{\delta(r - ct)}{4\pi c r} \int_{\partial B_r(\mathbf{x})} h(\sigma) d\sigma dr \\ &= \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\mathbf{x})} h(\sigma) d\sigma. \end{aligned}$$

where we have used the formula $\int_0^{\infty} \delta(r - ct) f(r) dr = f(ct)$.

– Lemma 1 and the above intuitive argument lead to the following theorem:

Theorem 2 (Kirchhoff's formula). *Let $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$. Then*

$$u(\mathbf{x}, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\mathbf{x})} g(\sigma) d\sigma\right] + \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\mathbf{x})} h(\sigma) d\sigma \tag{10}$$

is the unique solution $u \in C^2(\mathbb{R}^3 \times [0, +\infty))$ of problem (1).

Proof. Let $\sigma = \mathbf{x} + ct\omega$, where $\omega \in \partial B_1(\mathbf{0})$, we have $d\sigma = c^2 t^2 d\omega$ and we may write

$$w_g(\mathbf{x}, t) = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\mathbf{x})} g(\sigma) d\sigma = \frac{t}{4\pi} \int_{\partial B_1(\mathbf{0})} g(\mathbf{x} + ct\omega) d\omega.$$

Since $g \in C^3(\mathbb{R}^3)$, this formula shows that w_g satisfies the hypotheses of Lemma 1.

– Therefore, it is enough to check that

$$w_h(\mathbf{x},t) = \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(\mathbf{x})} h(\sigma)d\sigma = \frac{t}{4\pi}\int_{\partial B_1(\mathbf{0})} h(\mathbf{x}+ct\omega)d\omega.$$

solves problem (2). We have

$$\partial_t w_h(\mathbf{x},t) = \frac{1}{4\pi}\int_{\partial B_1(\mathbf{0})} h(\mathbf{x}+ct\omega)d\omega + \frac{ct}{4\pi}\int_{\partial B_1(\mathbf{0})} \nabla h(\mathbf{x}+ct\omega)\cdot\omega d\omega. \tag{11}$$

Thus

$$w_h(\mathbf{x},0) = 0 \text{ and } \partial_t w_h(\mathbf{x},0) = h(\mathbf{x}).$$

Moreover, by Gauss' formula, we may write

$$\begin{aligned}\frac{ct}{4\pi}\int_{\partial B_1(\mathbf{0})} \nabla h(\mathbf{x}+ct\omega)\cdot\omega d\omega =& \frac{1}{4\pi ct}\int_{\partial B_{ct}(\mathbf{x})} \partial_\nu h(\sigma)d\sigma \\ =& \frac{1}{4\pi ct}\int_{B_{ct}(\mathbf{x})} \Delta h(\mathbf{y})d\mathbf{y} \\ =& \frac{1}{4\pi ct}\int_0^{ct}\int_{\partial B_r(\mathbf{x})} \Delta h(\sigma)d\sigma dr,\end{aligned}$$

whence, from (11),

$$\begin{aligned}\partial_{tt} w_h(\mathbf{x},t) =& \frac{c}{4\pi}\int_{\partial B_1(\mathbf{0})} \nabla h(\mathbf{x}+ct\omega)\cdot\omega d\omega - \frac{1}{4\pi ct^2}\int_{B_{ct}(\mathbf{x})} \Delta h(\mathbf{y})d\mathbf{y} \\ &+ \frac{1}{4\pi t}\int_{\partial B_{ct}(\mathbf{x})} \Delta h(\sigma)d\sigma \\ =& \frac{1}{4\pi t}\int_{\partial B_{ct}(\mathbf{x})} \Delta h(\sigma)d\sigma.\end{aligned}$$

On the other hand,

$$\Delta w_h(\mathbf{x},t) = \frac{t}{4\pi}\int_{\partial B_1(\mathbf{0})} \Delta h(\mathbf{x}+ct\omega)d\omega = \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(\mathbf{x})} \Delta h(\sigma)d\sigma$$

and therefore $\partial_{tt} w_h - c^2\Delta w_h = 0$. □

Corollary. *Let $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$. Then*

$$u(\mathbf{x},t) = \frac{1}{4\pi c^2 t^2}\int_{\partial B_{ct}(\mathbf{x})} [g(\sigma) + \nabla g(\sigma)\cdot(\sigma-\mathbf{x}) + th(\sigma)]d\sigma \tag{12}$$

is the unique solution $u \in C^2(\mathbb{R}^3\times[0,+\infty))$ of problem (4).

- The presence of the gradient of g in (12) suggests that, unlike the one-dimensional case, the solution u **may be more irregular than the data**.

– Indeed, if $g \in C^k(\mathbb{R}^3)$ and $h \in C^{k-1}(\mathbb{R}^3)$, $k \geq 2$, then we can only guarantee that $u \in C^{k-1}$ and $u_t \in C^{k-2}$ at a later time.

- Formula (12) makes perfect sense also for $g \in C^1(\mathbb{R}^3)$ and h bounded.
  Clearly, under these weaker hypotheses, (12) satisfies the wave equation in an appropriate generalized sense.

- According to (12), $u(\mathbf{x}, t)$ depends on the data g and h only on the surface $\partial B_{ct}(\mathbf{x})$, which therefore coincides with **the domain of dependence for** $(\mathbf{x}, t)$.
- Assume that the support of g and h is the compact set D.
  Then $u(\mathbf{x}, t) \neq 0$ only for

$$t_{\min} < t < t_{\max},$$

where $t_{\min}$ and $t_{\max}$ are the **first** and the **last** time t such that $D \cap \partial B_{ct}(\mathbf{x}) \neq \emptyset$.

– In other words, a disturbance, initially localized inside D, starts affecting the point $\mathbf{x}$ at time $t_{\min}$, and ceases to affect it after time $t_{\max}$.

- Fix t and consider the union of all the spheres $\partial B_{ct}(\xi)$ as ξ varies on ∂D.
  The envelope of these surfaces constitutes the **wave front** and bounds the support of u, which spreads at speed c.

Another Approach: Spherical Mean Values

Definition. *Given $f = f(\mathbf{x}) \in C^2(\mathbb{R}^n)$, the function*

$$v = v(\mathbf{x}, r) = M(\mathbf{x}, r; f) = \frac{1}{\omega_n} \int_{|\xi|=1} f(\mathbf{x} + r\xi) d\sigma(\xi), \quad (\mathbf{x}, \tau) \in \mathbb{R}^n \times \mathbb{R}, \tag{13}$$

is denoted as the spherical integral mean-value of f over the sphere

$$\partial B_{|r|}(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : \ |\mathbf{y} - \mathbf{x}| = |r|\}.$$

Theorem 3 (F. John). *Given $f = f(\mathbf{x}) \in C^k(\mathbb{R}^n)$ with $k \geq 2$, the function $v = v(\mathbf{x}, r) = M(\mathbf{x}, r; f) : \mathbb{R}^n \to \mathbb{R}$ belongs to $C^k(\mathbb{R}^n \times \mathbb{R})$, and the following statements hold true:*

(a) $v(\mathbf{x}, 0) = f(\mathbf{x})$, *for all* $\mathbf{x} \in \mathbb{R}^n$,
(b) $v(\mathbf{x}, -r) = v(\mathbf{x}, r)$, *for all* $\mathbf{x} \in \mathbb{R}^n$, $r \in \mathbb{R}$,
(c) $\frac{\partial}{\partial r} v(\mathbf{x}, 0) = 0$, *for all* $\mathbf{x} \in \mathbb{R}^n$,
(d) $\frac{\partial^2}{\partial r^2} v(\mathbf{x}, r) + \frac{n-1}{r} \frac{\partial}{\partial r} v(\mathbf{x}, r) - \Delta_{\mathbf{x}} v(\mathbf{x}, r) = 0$ *in* $\mathbb{R}^n \times (\mathbb{R} \setminus \{0\})$.

We call the equation in (d) **Darboux's differential equation**.

Proof. (a) From (13) we infer $v \in C^k(\mathbb{R}^n \times \mathbb{R})$ and

$$v(\mathbf{x}, 0) = \frac{1}{\omega_n} \int_{|\xi|=1} f(\mathbf{x}) d\sigma(\xi) = f(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Also (13) implies (b) and (b) implies (c).
(d)

$$\begin{aligned}
\frac{\partial}{\partial r} v(\mathbf{x}, r) =& \frac{1}{\omega_n} \int_{\partial B_1(\mathbf{0})} \nabla f(\mathbf{x} + r\omega) \cdot \omega d\omega \\
=& \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\mathbf{x})} \partial_\nu f(\sigma) d\sigma \\
=& \frac{1}{\omega_n r^{n-1}} \int_{B_r(\mathbf{x})} \Delta f(\mathbf{y}) d\mathbf{y}, \quad \text{by the Gauss theorem,} \\
=& \frac{1}{\omega_n r^{n-1}} \int_0^r \int_{\partial B_s(\mathbf{x})} \Delta f(\sigma) d\sigma ds,
\end{aligned}$$

whence,

$$\begin{aligned}
\frac{\partial}{\partial r^2} v(\mathbf{x}, r) &= -\frac{n-1}{\omega_n r^n} \int_{B_r(\mathbf{x})} \Delta f(\mathbf{y}) d\mathbf{y} + \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\mathbf{x})} \Delta f(\mathbf{y}) d\sigma(\mathbf{y}) \\
&= -\frac{n-1}{r} \frac{\partial}{\partial r} v(\mathbf{x}, 0) + \frac{1}{\omega_n r^{n-1}} \Delta_{\mathbf{x}} \int_{\partial B_r(\mathbf{x})} f(\mathbf{y}) d\sigma(\mathbf{y}),
\end{aligned}$$

because

$$\begin{aligned}\Delta_{\mathbf{x}} \int_{\partial B_r(\mathbf{x})} f(\mathbf{y}) d\sigma(\mathbf{y}) =& \Delta_{\mathbf{x}} \int_{\partial B_r(\mathbf{x}_0)} f(\mathbf{x}-\mathbf{x}_0+\mathbf{y}) d\sigma(\mathbf{y}) \\ =& \int_{\partial B_r(\mathbf{x}_0)} \Delta_{\mathbf{x}} f(\mathbf{x}-\mathbf{x}_0+\mathbf{y}) d\sigma(\mathbf{y}) \\ =& \int_{\partial B_r(\mathbf{x})} \Delta f(\mathbf{y}) d\sigma(\mathbf{y}). \qquad \square\end{aligned}$$

- Specialize Theorem 3 to the case $n = 3$. The function

$$v(\mathbf{x}, r) = M(\mathbf{x}, r; h), \quad (\mathbf{x}, r) \in \mathbb{R}^3 \times (\mathbb{R} \setminus \{\mathbf{0}\})$$

satisfies Darboux's differential equation

$$\begin{aligned} 0 =& v_{rr}(\mathbf{x}, r) + \frac{2}{r} v_r(\mathbf{x}, r) - \Delta_{\mathbf{x}} v(\mathbf{x}, r) \\ =& \frac{1}{r}(rv(\mathbf{x}, r))_{rr} - \Delta_{\mathbf{x}} v(\mathbf{x}, r). \end{aligned}$$

Hence the function $rv(\mathbf{x}, r)$ satisfies

$$[rv(\mathbf{x}, r)]_{rr} - \Delta_{\mathbf{x}}[rv(\mathbf{x}, r)] = 0, \quad (\mathbf{x}, r) \in \mathbb{R}^3 \times \mathbb{R}.$$

We now consider the function

$$\psi(\mathbf{x}, t) = \frac{1}{c}[ctv(\mathbf{x}, ct)] = tv(\mathbf{x}, ct) = \frac{t}{4\pi} \int\limits_{|\xi|=1} h(\mathbf{x} + ct\xi) d\sigma(\xi)$$

with $(\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}$. It fulfills the wave equation

$$\frac{\partial^2}{\partial t^2}\psi(\mathbf{x}, t) - c^2 \Delta \psi(\mathbf{x}, t) = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}$$

and is subject to the initial conditions

$$\psi(\mathbf{x}, 0) = 0, \quad \frac{\partial}{\partial t}\psi(\mathbf{x}, 0) = v(\mathbf{x}, 0) = h(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^3.$$

- To solve the Cauchy problem

$$(14) \qquad \begin{cases} u_{tt} - c^2 \Delta u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R} \\ u(\mathbf{x}, 0) = g(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = h(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

we observe that the function

$$\chi(\mathbf{x}, t) = tM(\mathbf{x}, ct; g) = \frac{t}{4\pi} \int\limits_{|\xi|=1} g(\mathbf{x} + ct\xi) d\sigma(\xi)$$

also satisfies the wave equation

$$\frac{\partial^2}{\partial t^2}\chi(\mathbf{x},t) - c^2\Delta\chi(\mathbf{x},t) = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}$$

Furthermore, we have $\chi \in C^3(\mathbb{R}^3 \times \mathbb{R})$. Then we consider the function

$$\varphi(\mathbf{x},t) = \frac{\partial}{\partial t}\chi(\mathbf{x},t).$$

The function φ also satisfies the wave equation and we have the initial conditions

$$\varphi(\mathbf{x},0) = M(\mathbf{x},0;g) = g(\mathbf{x}),$$

$$\frac{\partial}{\partial t}\varphi(\mathbf{x},0) = \frac{\partial^2}{\partial t^2}\chi(\mathbf{x},0) = c^2\Delta_{\mathbf{x}}\chi(\mathbf{x},0) = 0, \quad \forall x \in \mathbb{R}^3.$$

- We thus obtain a solution of the problem (14)

$$\begin{aligned} u(\mathbf{x},t) =& \varphi(\mathbf{x},t) + \psi(\mathbf{x},t) \\ =& \frac{\partial}{\partial t}\Big\{tM(\mathbf{x},ct;g)\Big\} + tM(\mathbf{x},ct;h), \end{aligned}$$

in which

$$\begin{aligned} \frac{\partial}{\partial t}\Big\{tM(\mathbf{x},ct;g)\Big\} =& M(\mathbf{x},ct;g) + t\frac{\partial}{\partial t}M(\mathbf{x},ct;g) \\ =& \frac{t}{4\pi}\int\limits_{|\xi|=1} g(\mathbf{x}+ct\xi)d\sigma(\xi) + \frac{t}{4\pi}\frac{\partial}{\partial t}\Big\{\int\limits_{|\xi|=1} g(\mathbf{x}+ct\xi)d\sigma(\xi)\Big\} \\ =& \frac{1}{4\pi}\Big\{\int\limits_{|\xi|=1} g(\mathbf{x}+ct\xi) + ct\nabla g(\mathbf{x}+ct\xi)\cdot\xi\Big\}d\sigma(\xi). \end{aligned}$$

With the aid of the substitution

$$\mathbf{y} = \mathbf{x} + ct\xi, \quad d\sigma(\mathbf{y}) = c^2t^2 d\sigma(\xi)$$

we deduce the following identity for all $(\mathbf{x},t) \in \mathbb{R}^3 \times \mathbb{R}$,

$$u(\mathbf{x},t) = \frac{1}{4\pi c^2t^2}\Big\{\int\limits_{|\mathbf{x}-\mathbf{y}|=ct} g(\mathbf{y}) + \nabla g(\mathbf{y})\cdot(\mathbf{y}-\mathbf{x}) + th(\mathbf{y})\Big\}d\sigma(\mathbf{y}).$$

Theorem 4 (Kirchhoff's formula). *Let $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$. Then*

(15)

$$\begin{aligned} u(\mathbf{x},t) =& \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(\mathbf{x})} g(\mathbf{y})d\sigma(\mathbf{y})\right] + \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(\mathbf{x})} h(\mathbf{y})d\sigma(\mathbf{y}) \\ =& \frac{1}{4\pi c^2t^2}\int_{\partial B_{ct}(\mathbf{x})}[g(\mathbf{y}) + \nabla g(\mathbf{y})\cdot(\mathbf{y}-\mathbf{x}) + th(\mathbf{y})]d\sigma(\mathbf{y}) \end{aligned}$$

is the unique solution $u \in C^2(\mathbb{R}^3 \times [0,+\infty))$ of problem (14).

Cauchy Problem in dimension 2

- The solution of the Cauchy problem in two dimensions can be obtained from Kirchhoff's formula, using the so-called **Hadamard's method of descend**.
- Consider first the problem

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, & \mathbf{x} \in \mathbb{R}^2, \quad t > 0, \\ u(\mathbf{x}, 0) = 0, \quad u_t(\mathbf{x}, 0) = h(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^2. \end{cases} \tag{16}$$

The key idea is to "immerse" the two-dimensional problem (16) in a three-dimensional setting.

– More precisely, write points in $\mathbb{R}^3$ as $(\mathbf{x}, x_3)$ and set $h(\mathbf{x}, x_3) = h(\mathbf{x})$.
The solution U of the three-dimensional problem is given by Kirchhoff's formula

$$U(\mathbf{x}, x_3, t) = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\mathbf{x}, x_3)} h(\sigma) d\sigma. \tag{17}$$

Claim: since h does not depend on x_3, U **is also independent of** x_3, and therefore the solution of (16) is given by (17) with, say, $x_3 = 0$.

– To prove the claim, note that the spherical surface $\partial B_{ct}(\mathbf{x}, x_3)$ is a union of the two hemispheres whose equations are

$$y_3 = F_{\pm}(y_1, y_2) = x_3 \pm \sqrt{c^2 t^2 - r^2}, \quad \text{where } r^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2.$$

On both hemispheres we have

$$\begin{aligned} d\sigma =& \sqrt{1 + |\nabla F_{\pm}|^2} dy_1 dy_2 \\ =& \sqrt{1 + \frac{r^2}{c^2 t^2 - r^2}} dy_1 dy_2 = \frac{ct}{\sqrt{c^2 t^2 - r^2}} dy_1 dy_2 \end{aligned}$$

so that we may write ($d\mathbf{y} = dy_1 dy_2$)

$$U(\mathbf{x}, x_3, t) = \frac{1}{2\pi c} \int_{B_{ct}(\mathbf{x})} \frac{h(\mathbf{y})}{\sqrt{c^2 t^2 - |\mathbf{x} - \mathbf{y}|^2}} d\mathbf{y}$$

and U is independent of x_3 as claimed.

- From the above calculations and recalling the uniqueness of the solution to the Cauchy problem, we deduce the following theorem.

Theorem 5 (Poisson's formula). *Let $g \in C^3(\mathbb{R}^2)$ and $h \in C^2(\mathbb{R}^2)$. Then*

$$u(\mathbf{x}, t) = \frac{1}{2\pi c} \left\{ \frac{\partial}{\partial t} \left(\int_{B_{ct}(\mathbf{x})} \frac{g(\mathbf{y})}{\sqrt{c^2 t^2 - |\mathbf{x} - \mathbf{y}|^2}} d\mathbf{y} \right) + \int_{B_{ct}(\mathbf{x})} \frac{h(\mathbf{y})}{\sqrt{c^2 t^2 - |\mathbf{x} - \mathbf{y}|^2}} d\mathbf{y} \right\}.$$

is the unique solution in $C^2(\mathbb{R}^2 \times [0, +\infty))$ of the problem

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, & \mathbf{x} \in \mathbb{R}^2, \quad t > 0, \\ u(\mathbf{x}, 0) = g(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = h(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^2. \end{cases}$$

- Also Poisson's formula can be written in a somewhat more explicit form. Indeed, letting $\mathbf{y}-\mathbf{x}=ct\mathbf{z}$, we have

$$d\mathbf{y}=c^2t^2d\mathbf{z}, \quad |\mathbf{x}-\mathbf{y}|^2=c^2t^2|\mathbf{z}|^2$$

whence

$$\int_{B_{ct}(\mathbf{x})}\frac{g(\mathbf{y})}{\sqrt{c^2t^2-|\mathbf{x}-\mathbf{y}|^2}}d\mathbf{y}=ct\int_{B_1(\mathbf{0})}\frac{g(\mathbf{x}+ct\mathbf{z})}{\sqrt{1-|\mathbf{z}|^2}}d\mathbf{z}.$$

Then

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{B_{ct}(\mathbf{x})}\frac{g(\mathbf{y})}{\sqrt{c^2t^2-|\mathbf{x}-\mathbf{y}|^2}}d\mathbf{y}\\ =&c\int_{B_1(\mathbf{0})}\frac{g(\mathbf{x}+ct\mathbf{z})}{\sqrt{1-|\mathbf{z}|^2}}d\mathbf{z}+c^2t\int_{B_1(\mathbf{0})}\frac{\nabla g(\mathbf{x}+ct\mathbf{z})\cdot\mathbf{z}}{\sqrt{1-|\mathbf{z}|^2}}d\mathbf{z}\end{aligned}$$

and going back to the original variables, we obtain

$$u(\mathbf{x},t)=\frac{1}{2\pi ct}\int_{B_{ct}(\mathbf{x})}\frac{g(\mathbf{y})+\nabla g(\mathbf{y})\cdot(\mathbf{y}-\mathbf{x})+th(\mathbf{y})}{\sqrt{c^2t^2-|\mathbf{x}-\mathbf{y}|^2}}d\mathbf{y}.$$

- Poisson's formula displays an important difference with respect to its three-dimensional analogue, Kirchhoff's formula.

– In fact the **domain of dependence** for the point $(\mathbf{x},t)$ is given by the **full circle**

$$B_{ct}(\mathbf{x})=\{\mathbf{y}:|\mathbf{x}-\mathbf{y}|<ct\}.$$

– This entails that a disturbance, initially localized at ξ, starts affecting the point $\mathbf{x}$ at time $t_{\min}=|\mathbf{x}-\xi|/c$.
However, **this effect does not vanish for** $t>t_{\min}$, since ξ still belongs to the disk $B_{ct}(\mathbf{x})$ after $t_{\min}$.

– It is the phenomenon one way observe by placing a cork on still water and dropping a stone not too far away.

- The cock remains undisturbed until it is reached by the wave front but its oscillations persist thereafter.

– Thus, sharp signals do not exist in dimensions two and **the strong Huygens principle does not hold.**

- An examination of Poisson's formula reveals the following.

Proposition. *The fundamental solution for the two dimensional wave equation is given by*

$$K(\mathbf{x},\mathbf{y},t)=\frac{1}{2\pi c}\frac{\mathcal{H}(ct-r)}{\sqrt{c^2t^2-r^2}}$$

where $r=|\mathbf{x}-\mathbf{y}|$ *and* $\mathcal{H}$ *is the Heaviside function. For* $\mathbf{y}$ *fixed, its support is the* **full** *forward space-time cone, with vertex at* $(\mathbf{y},0)$ *and opening* $\theta=\tan^{-1}c$*, given by*

$$C^*_{\mathbf{y},0}=\{(\mathbf{x},t):|\mathbf{x}-\mathbf{y}|\le ct,\ t>0\}.$$

Nonhomogeneous Equation. Retarded potential.

- Consider the nonhomogeneous problem

$$(18) \qquad \begin{cases} u_{tt} - c^2\Delta u = f(\mathbf{x}, t), & \mathbf{x} \in \mathbb{R}^3, \quad t > 0, \\ u(\mathbf{x}, 0) = 0, \quad u_t(\mathbf{x}, 0) = 0, & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

  Assume that $f \in C^2(\mathbb{R}^3 \times [0, +\infty))$.

Step (i) Consider the homogeneous Cauchy problem

$$\begin{cases} w_{tt} - c^2\Delta w = 0, & \mathbf{x} \in \mathbb{R}^3, \quad t > 0 \\ w(\mathbf{x}, s; s) = 0, \quad w_t(\mathbf{x}, s; s) = f(\mathbf{x}, s), & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

– Since the wave equation is invariant under time translations, the solution of (19) is given by Kirchhoff's formula with t replaced by $t - s$:

$$w(\mathbf{x}, t; s) = \frac{1}{4\pi c^2(t-s)} \int_{\partial B_{c(t-s)}(\mathbf{x})} f(\sigma, s) d\sigma.$$

Step (ii) Integrating w over $(0, t)$ with respect to s, then

$$(19) \qquad u(\mathbf{x}, t) = \int_0^t w(\mathbf{x}, t; s) ds = \frac{1}{4\pi c^2} \int_0^t \frac{1}{(t-s)} \int_{\partial B_{c(t-s)}(\mathbf{x})} f(\sigma, s) d\sigma ds$$

is the unique solution $u \in C^2(\mathbb{R}^3) \times [0, +\infty)$ of (18).

Remark. Formula (19) shows that the value of u at the point $(\mathbf{x}, t)$ depends on the values of f in the full **backward cone**

$$C_{\mathbf{x},t} = \{(\mathbf{z}, s) : |\mathbf{z} - \mathbf{x}| \leq c(t-s), \ 0 \leq s \leq t\}.$$

Note that (19) can be written in the form

$$u(\mathbf{x}, t) = \frac{1}{4\pi} \int_{B_{ct}(\mathbf{x})} \frac{1}{|\mathbf{x} - \mathbf{y}|} f\left(\mathbf{y}, t - \frac{|\mathbf{x} - \mathbf{y}|}{c}\right) d\mathbf{y}$$

which is the so called **retarded potential**. Indeed, $u(\mathbf{x}, t)$ depends on the values of the source $f(\mathbf{y})$ at the earlier times

$$t' = t - \frac{|\mathbf{x} - \mathbf{y}|}{c}.$$

- The solution of the two-dimensional non-homogeneous Cauchy problem with zero initial data

$$\begin{cases} u_{tt} - c^2\Delta u = f(\mathbf{x}, t), & \mathbf{x} \in \mathbb{R}^2, \quad t > 0, \\ u(\mathbf{x}, 0) = 0, \quad u_t(\mathbf{x}, 0) = 0, & x \in \mathbb{R}^2 \end{cases}$$

  is given by

$$u(\mathbf{x}, t) = \frac{1}{4\pi c} \int_0^t \int_{B_{c(t-s)}(\mathbf{x})} \frac{1}{\sqrt{c^2(t-s)^2 - |\mathbf{x} - \mathbf{y}|^2}} f(\mathbf{y}, s) d\mathbf{y} ds.$$