The Method of Characteristics for Quasilinear Equations

- Recall a simple fact from the theory of ODE's:
  The equation $\frac{du}{dt} = f(t, u)$ can be solved (at least for small values of t) for each initial condition $u(0) = u_0$, provided that f is continuous in t and Lipschitz continuous in the variable u.
  Recall that the solution may exist globally in time, or may blow up at some finite time.

- If we allow the equation and the initial condition to depend on a parameter x, then the solution u depends on x and may be written as $u(x, t)$.

– In fact, u becomes a solution of

$$\begin{cases} u_t = f(x, t, u), \\ u(x, 0) = u_0(x), \end{cases}$$

  that may be thought of as an initial value problem for a PDE in which u_x does not appear.

– Assuming f and u_0 are continuous functions of x, the solution $u(x, t)$ will be continuous in x (and t).

⊙ **Geometrically**, the graph $z = u(x, t)$ is a surface in $\mathbb{R}^3$ that contains the curve $(x, 0, u_0(x))$.

– This surface may be defined for all $t > 0$, or may blow up at some finite t_0 (which may depend on x).

- However, if the surface remains bounded, then it will continue as a graph for all $t > 0$.

- In particular, the surface **cannot** fold over on itself and thereby fail to be the graph of a function.

- These elementary ideas from ODE theory lie behind the method of characteristics which applies to general quasilinear first-order PDE's, as we shall discover in this section.

Example (The Transport Equation). Consider the initial value problem for the transport equation

$$\begin{cases} u_t + au_x = 0, \\ u(x, 0) = h(x), \end{cases}, \quad \text{where } a \text{ is a constant.}$$

Reduce this problem to an ODE along some curve $x(t)$ by finding $x(t)$ so that

$$\frac{d}{dt}u(x(t), t) = au_x + u_t.$$

By the chain rule, we simply require $\frac{dx}{dt} = a$, i.e.,

$$x = at + x_0, \quad \text{where } x_0 \text{ is the } x\text{-intercept of the curve..}$$

Along this curve we have $u_t = 0$, i.e. $u \equiv \text{constant}\, h(x_0) = h(x - at)$.

– Indeed, if h is C^1, then we can check that $u(x, t) = h(x - at)$ satisfies the PDE and the initial condition.

– The lines $x = at + x_0$ are called the **characteristic curves** of $u_t + au_x = 0$.

- The reduction of a PDE to an ODE along its characteristics is called the **method of characteristics**, and applies to much more complicated equations.

– Let us now see how and why this method also applies to quasilinear PDEs.

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1. Characteristics.

- We consider the quasilinear equation equations of the form

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u), \tag{1}$$

where $u = u(x,y)$ is a continuously differentiable function of the two variables x and y and a, b, c are continuously differentiable functions of three variables x, y and u.

- The solutions of (1) can be constructed via geometric arguments.

– Namely, if $u(x,y)$ is a solution of (1), let us consider its graph $z = u(x,y)$.

– The tangent plane to the graph of a solution u at a point (x_0, y_0, z_0) has equation

$$u_x(x_0,y_0)(x-x_0) + u_y(x,y)(y-y_0) - (z-z_0) = 0$$

and the vector

$$\mathbf{n}_0 = (u_x(x_0,y_0), u_y(x_0,y_0), -1)$$

is normal to the plane. Introducing the vector

$$\mathbf{v}_0 = (a(x_0,y_0,u), b(x_0,y_0,u), c(x_0,y_0,u)),$$

then the equation (1) implies that

$$\mathbf{n}_0 \cdot \mathbf{v}_0 = 0.$$

Thus, $\mathbf{v}_0$ is tangent to the graph of u, and thus must lie on the tangent plane to the graph of $z = u(x,y)$ at the point (x_0, y_0, z_0).

– In other words, (1) says that

$$\mathbf{v}(x,y,z) = (a(x,y,u), b(x,y,u), c(x,y,u))$$

defines a vector field in $\mathbb{R}^3$, to which graphs of solutions must be tangent at each point (x,y,z).

Definition. *Surfaces which are tangent at each point to a vector field in $\mathbb{R}^3$ are called* **integral surfaces** *of the vector field.*
Curves which are tangent at each point to a vector field in $\mathbb{R}^3$ are called **integral curves** *of the vector field.*

– Thus to find a solution of (1), we should try to find integral surfaces of $\mathbf{v}$.

- How can we construct integral surfaces?
We may construct integral surfaces of $\mathbf{v}$ as union of **integral curves of v**, that is curves tangent to $\mathbf{v}$ at every point.These curves are solutions of the system

$$\frac{dx}{dt} = a(x,y,z), \quad \frac{dy}{dt} = b(x,y,z), \quad \frac{dz}{dt} = c(x,y,z) \tag{2}$$

and are called **characteristics**. Note that $z = z(t)$ gives the values of u along a characteristic; that is,

$$z(t) = u(x(t), y(t)). \tag{3}$$

In fact, differentiating (3) and using (2) and (1), we have

$$\begin{aligned} c(x(t),y(t),z(t)) =& \frac{dz}{dt} = u_x(x(t),y(t))\frac{dx}{dt} + u_y(x(t),y(t))\frac{dy}{dt} \\ =& a(x(t),y(t),z(t))u_x(x(t),y(t)) + b(x(t),y(t),z(t))u_y(x(t),y(t)). \end{aligned}$$

Thus, along a characteristic the PDE (1) degenerates into an ODE.

- The following proposition is a consequence of the above geometric reasoning and of the existence and uniqueness theorem for system of ODE's.

Proposition 1. *(a) Let the surface S be the graph of a C^1 function $u = u(x, y)$. If S is a union of characteristics, then u is a solution of the equation (1); (in other words,* **a smooth union of characteristic curves is an integral surfaces***).*

(b) Every integral surface S of the vector field $\mathbf{v}$ is a union of characteristics. Namely, every point of S belongs exactly to one characteristic, entirely contained in S.

(c) Two integral surfaces intersecting at one point intersect along the whole characteristic passing through the point.

- In the case of a conservation law (with $t = y$)

$$u_y + q'(u)u_x = 0 \qquad \left(q'(u) = \frac{dq}{du}\right),$$

we have introduced the notion of characteristics in a slightly different way, but we see below that there is no contradiction.

- **Conservation Laws.** According to the new definition, the characteristics of the equation

$$u_y + q'(u)u_x = 0 \qquad \left(q'(u) = \frac{dq}{du}\right),$$

with initial conditions

$$u(x, 0) = g(x)$$

are the three-dimensional solution curves of the system

$$\frac{dx}{dt} = q'(z), \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 0$$

with initial conditions

$$x(s, 0) = s, \quad y(s, 0) = 0, \quad z(s, 0) = g(s), \quad s \in \mathbb{R}.$$

Integrating, we find

$$z = g(s), \quad x = q'(g(s))t + s, \quad y = t.$$

The **projections** of these straight lines on the (x, y)-plane are

$$x = q'(g(s))y + s,$$

which are the "old characteristics", called **projected characteristics** in the general quasilinear context.

- **Linear Equations.** Consider the linear equation

$$a(x, y)u_x + b(x, y)u_y = 0.$$

Introducing the vector $\mathbf{w} = (a, b)$, we may write this equation in the form

$$D_{\mathbf{w}}u = \nabla u \cdot \mathbf{w} = 0.$$

Thus, every solution u is constant along the integral lines of the vector $\mathbf{w}$, i.e. along the **projected characteristics**, which are solutions of the reduced characteristic system

$$\frac{dx}{dt} = a(x, y), \quad \frac{dy}{dt} = b(x, y),$$

locally equivalent to the ODE $b(x, y)dx - a(x, y)dy = 0$.

The Cauchy Problem:

- Proposition 1 gives a characterization of the integral surfaces as a union of characteristics.
- The problem is how to construct such unions to obtain a **smooth surface**.
- One way to proceed is to proceed is to look for solutions u **whose values are prescribed on a curve γ_0, contained in the (x, y)-plane.**

- In other words, suppose that

$$x(s) = f(s), \quad y(s) = g(s), \quad s \in I \subset \mathbb{R}$$

is a parametrization of γ_0. We look for a solution u of (1) such that

$$u(f(s), g(s)) = h(s), \qquad s \in I, \tag{4}$$

where $h = h(s)$ is a given function.
We assume that I is a neighborhood of $s = 0$, and that f, g, h are in $C^1(I)$.

- The system (1), (4) is called **Cauchy problem**.
- If we consider the three-dimensional curve Γ_0 given by the parametrization

$$x(s) = f(s), \quad y(s) = g(s), \quad z(s) = h(s),$$

then, solving the Cauchy problem (1), (4) amounts to **determining an integral surface containing** Γ_0.

The Cauchy Problem: Given a curve Γ_0 in $\mathbb{R}^3$, can we find a solution u of (1) whose graph contains Γ_0?

- In the special case that Γ_0 is the graph $(x, h(x))$ in the xz-plane of a function h, the Cauchy problem is just an initial value problem with the obvious interpretation of the variable y as “time”.
- Namely, the data are assigned in the form of **initial values**

$$u(x, 0) = h(x),$$

with y playing the role of “time”. In this case, γ_0 is the axis $y = 0$ and x plays the role of the parameter s.
Then a parametrization of Γ_0 is given by

$$x = x, \quad y = 0, \quad z(x) = h(x).$$

- By analogy, we often refer to Γ_0 as the **initial curve**.
- The strategy to solve a Cauchy problem comes from its geometric meaning: since the graph of a solution $u = u(x, y)$ is a smooth union of characteristics, we flow out from each point of Γ_0 along the characteristic curve through that point, thereby sweeping out an integral surface, which is the union of the characteristics and should give the graph of u.
This is the **method of characteristics**.

- Actually, this construction of an integral surface containing Γ_0 can be achieved by writing Γ_0 as the graph of a curve

$$x(0) = f(s), \quad y(0) = g(s), \quad z(0) = h(s), \tag{5}$$

parameterized by $s \in I$, and then for each s solving the system

$$\frac{dx}{dt} = a(x, y, z), \quad \frac{dy}{dt} = b(x, y, z), \quad \frac{dz}{dt} = c(x, y, z) \tag{6}$$

using (5) as initial conditions.
- Under our hypotheses, the Cauchy problem (5), (6) has a unique solution

$$x = X(s, t), \quad y = Y(s, t), \quad z = Z(s, t), \tag{7}$$

in a neighborhood of $t = 0$, for every $s \in I$.
- In this way, we obtain our integral surface parameterized by s and t.
- To find the solution u of (1), it remains only to **replace the variables** s **ant** t **by expressions involving** x **and** y.

- Thus, a couple of questions arise:
  (a) Do the three equations (7) define a function $z = u(x, y)$?
  (b) Even if the answer to **(a)** is positive,
  is $z = u(x, y)$ the **unique** solution of the Cauchy problem?

- Let us reason in a neighborhood of $s = t = 0$, setting

$$X(0, 0) = f(0) = x_0, \quad Y(0, 0) = g(0) = y_0, \quad Z(0, 0) = h(0) = z_0.$$

(a) The answer to question **(a)** is positive if we can solve for s and t in terms of x and y via the first two equations in (7), and find $s = S(x, y)$, $t = T(x, y)$ of class C^1 in a neighborhood of (x_0, y_0), such that

$$S(x_0, y_0) = 0, \quad T(x_0, y_0) = 0;$$

then, from the third equation $z = Z(s, t)$, we obtain

$$z = Z(S(x, y), T(x, y)) = u(x, y). \tag{8}$$

From the **Inverse Function Theorem**, the system

$$\begin{cases} X(s, t) = x \\ Y(s, t) = y \end{cases}$$

defines

$$s = S(x, y) \quad \text{and} \quad t = T(x, y)$$

in a neighborhood of (x_0, y_0) if

$$J(0, 0) = \begin{vmatrix} X_s(0, 0) & Y_s(0, 0) \\ X_t(0, 0) & Y_t(0, 0) \end{vmatrix} \neq 0. \tag{9}$$

From (5) and (6), we have

$$X_s(0,0) = f'(0), \quad Y_s(0,0) = g'(0)$$

and

$$X_t(0,0) = a(x_0, y_0, z_0), \quad Y_t(0,0) = b(x_0, y_0, z_0),$$

so that (9) becomes

$$J(0,0) = \begin{vmatrix} f'(0) & g'(0) \\ a(x_0, y_0, z_0) & b(x_0, y_0, z_0) \end{vmatrix} \neq 0. \tag{10}$$

or equivalently,

$$b(x_0, y_0, z_0) f'(0) \neq a(x_0, y_0, z_0) g'(0). \tag{11}$$

Geometrically, condition (11) means that the vectors

$$(a(x_0, y_0, z_0), b(x_0, y_0, z_0)) \quad \text{and} \quad (f'(0), g'(0))$$

are **not parallel**; in other words, the tangent to Γ_0 and the vector field (a, b, c) along Γ_0 project to vectors in the xy-plane which are nowhere parallel.

- **Conclusion**: if condition (10) holds, then (8) is a well-defined C^1-function in a neighborhood of (x_0, y_0).

(b) The above consideration of u implies that the surface $z = u(x, y)$ contains Γ_0 and all the characteristics flowing out from Γ_0, so that u is a solution of the Cauchy problem.

– Moreover, by Proposition 1 (c), two integral surfaces containing Γ_0 must contain the same characteristics and therefore coincide.

- We summarize everything in the following, recalling that

$$(x_0, y_0, z_0) = (f(0), g(0), h(0)).$$

Theorem 2. *Let a, b, c be C^1-functions in a neighborhood of (x_0, y_0, z_0) and f, g, h be C^1-functions in I. If*

$$J(0,0) \neq 0,$$

then, in a neighborhood of (x_0, y_0), there exists a unique C^1-solution $u = u(x, y)$ of the Cauchy problem

$$\begin{cases} a(x, y, u) u_x + b(x, y, u) u_y = c(x, y, u) \\ u(f(s), g(s)) = h(s). \end{cases} \tag{12}$$

Moreover, u is defined by the parametric equation (7).

Remark. If a, b, c and f, g, h are C^k-functions, $k \geq 2$, then u is also a C^k-function.

- It remains to **examine what happens when** $J(0,0) = 0$, that is when the vectors $(a(x_0, y_0, z_0), b(x_0, y_0, z_0))$ and $(f'(0), g'(0))$ are parallel.

⊙ Suppose that there exists a C^1-solution u of the Cauchy problem (12).
Differentiating the second equation in (12), we obtain

$$h'(s) = u_x(f(s), g(s))f'(s) + u_y(f(s), g(s))g'(s). \tag{13}$$

Computing at $x = x_0$, $y = y_0$, $z = z_0$ and $s = 0$, we obtain

$$\begin{cases} a(x_0, y_0, u_0)u_x(x_0, y_0) + b(x_0, y_0, u_0)u_y(x_0, y_0) = c(x_0, y_0, u_0) \\ f'(0)u_x(x_0, y_0) + g'(0)u_y(x_0, y_0)g'(0) = h'(0). \end{cases} \tag{14}$$

– Since u is a solution of the Cauchy problem, the vector $(u_x(x_0, y_0), u_y(x_0, y_0))$ is a solution of the algebraic system (14).
– But then we know that, if $J(0, 0) = 0$, the condition

$$\operatorname{rank}\begin{pmatrix} a(x_0, y_0, z_0) & b(x_0, y_0, z_0) & c(x_0, y_0, z_0) \\ f'(0) & g'(0) & h'(0) \end{pmatrix} = 1 \tag{15}$$

must hold; i.e., the two vectors

$$(a(x_0, y_0, z_0), b(x_0, y_0, z_0), c(x_0, y_0, z_0)) \quad \text{and} \quad (f'(0), g'(0), h'(0))$$

must be parallel. This is equivalent to saying that Γ_0 is parallel to the characteristic curve at (x_0, y_0, z_0).
When this occurs, we say that Γ_0 **is characteristic at the point** (x_0, y_0, z_0).

Conclusion. *If $J(0, 0) = 0$, a necessary condition for the existence of a C^1-solution $u = u(x, y)$ of the Cauchy problem in a neighborhood of (x_0, y_0) is that Γ_0 be characteristic at (x_0, y_0, z_0).*

- Now assume that Γ_0 itself is a characteristic and let $P_0 = (x_0, y_0, z_0) \in \Gamma_0$.

– If we choose a curve Γ^* transversal to Γ_0 at P_0,
by Theorem 2 there exists a unique integral surface containing Γ^*
and, by Proposition 1 (c), this surface containing Γ_0.
– In this way we can construct infinitely many solutions.

- We point out that the condition (15) is compatible with the existence of a C^1-solution only if Γ_0 is characteristic at P_0.
- On the other hand, it may occur that Γ_0 is noncharacteristic at P_0 and that the solutions of the Cauchy problem exist anyway; clearly, these solutions **cannot be of class C^1**.

- Let us summarize the steps to solve the Cauchy problem (12):

Step 1. Determine the solution (7) of the characteristic system (5) with initial conditions

$$X(s, 0) = f(s), \quad Y(s, 0) = g(s), \quad Z(s, 0) = h(s), \quad s \in I.$$

Step 2. Compute $J(s,t)$ on the initial curve Γ_0; i.e.

$$J(s,0) = \begin{vmatrix} f'(s)) & g'(s) \\ X_t(s,0) & Y_t(s,0) \end{vmatrix}.$$

The following cases may occur:

Case 2a. $J(s,0) \neq 0$ **for every** $s \in I$; i.e. Γ_0 does not have characteristic points. Then, in a neighborhood of Γ_0, there exists a unique solution $u = u(x,y)$ of the Cauchy problem, defined by the parametric equation (7).

Case 2b. $J(s_0,0) = 0$ **for some** $s_0 \in I$ **and** Γ_0 **is characteristic at the point** $P_0 = (f(s_0), g(s_0), h(s_0))$.
A C^1-solution may exist in a neighborhood of P_0 only if the rank condition (15) holds at P_0.

Case 2c. $J(s_0,0) = 0$ **for some** $s_0 \in I$ **and** Γ_0 **is** *not* **characteristic at** P_0. There are no C^1-solutions in a neighborhood of P_0. There may exist less regular solutions.

Case 2d. Γ_0 **is a characteristic**. Then there exists infinitely many C^1-solutions in a neighborhood of Γ_0.

Example 1. Consider the nonhomogeneous Burger equation

$$uu_x + u_y = 1 \tag{16}$$

If y is the time variable, then $u = u(x,y)$ represents a *velocity field* of a flux of particles along the x-axis. Equation (16) states that the acceleration of each particle is equal to 1. Assume

$$u(x,0) = h(x), \quad x \in \mathbb{R}.$$

The characteristics are solutions of the system

$$\frac{dx}{dt} = z, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 1$$

and the initial curve Γ_0 has the parametrization

$$x = f(s) = s, \quad y = g(s) = 0, \quad z = h(s), \quad s \in \mathbb{R}.$$

The characteristics flowing out from Γ_0 are

$$X(s,t) = s + \frac{t^2}{2} + th(s), \quad Y(s,t) = t, \quad Z(s,t) = t + h(s), \quad s \in \mathbb{R}.$$

Since

$$J(s,t) = \begin{vmatrix} 1 + th'(s) & 0 \\ t + h(s) & 1 \end{vmatrix} = 1 + th'(s),$$

we have $J(s,0) = 1$ and we are in **Case 2a**: in a neighborhood of Γ_0 there exists a unique C^1-solution.

– If, for instance, $h(s) = s$, we find the solution

$$u = y + \frac{2x - y^2}{2 + 2y}, \quad \forall x \in \mathbb{R}, \quad y \geq -1.$$

⊙ Now consider the same equation with initial condition

$$u\left(\frac{y^2}{4}, y\right) = \frac{y}{2},$$

assigning the values of u on the parabola $x = \frac{y^2}{4}$. A parametrization of Γ_0 is given by

$$x = s^2, \quad y = 2s, \quad z = s, \quad s \in \mathbb{R}.$$

Solving the characteristic system with these initial conditions, we find

$$(17) \qquad X(s,t) = s^2 + ts + \frac{t^2}{2}, \quad Y(s,t) = 2s + t, \quad Z(s,t) = s + t, \quad s \in \mathbb{R}.$$

– Observe that Γ_0 does not have any characteristic point, since its tangent vector $(2s, 2, 1)$ is never parallel to the characteristic direction $(s, 1, 1)$. However

$$J(s,t) = \begin{vmatrix} 2s+t & 2 \\ s+t & 1 \end{vmatrix} = -t,$$

which vanishes for $t = 0$, i.e. exactly on Γ_0. We are in **Case 2c**.

– Solving for s and t, $t \neq 0$, in the first two equations (17), and substituting into the third one, we find

$$u(x,y) = \frac{y}{2} \pm \sqrt{x - \frac{y^2}{4}}.$$

We have found two solutions of the Cauchy problem, satisfying the differential equation in the region $x > \frac{y^2}{4}$. However, these solutions are **not** smooth in a neighborhood of Γ_0, since on Γ_0 they are **not** differentiable.

- If Γ satisfies (9), the solution may develop singularities away from Γ (i.e. for larger values of t).

– Geometrically, this may be due to the integral surface folding over on itself at some point (x_1, y_1).

– In such a case, the solution experiences a "gradient catastrophe" (i.e. u_x becomes infinite) as $(x,y) \to (x_1, y_1)$, and therefore the solution u cannot be both single-valued and continuous.

Example 2. Consider the initial value problem

$$uu_x + yu_y = x$$

with

$$u(x,1) = 2x, \quad x \in \mathbb{R}.$$

The characteristics are solutions of the system

$$\frac{dx}{dt} = z, \quad \frac{dy}{dt} = y, \quad \frac{dz}{dt} = x$$

and the initial curve Γ_0 has the parametrization

$$x = f(s) = s, \quad y = g(s) = 1, \quad z = 2s, \quad s \in \mathbb{R}.$$

We easily checked that (9) is satisfied: $1 \neq 0$.

- Note that the characteristic equation for y happens to be decouple and may be integrated to obtain $y = c(s)e^t$, and the initial condition then yields $y = e^t$.
- The equations for x and z form a 2×2 system, which may be solved by finding eigenvalues and eigenvectors, or we can simply observe that

  $$\frac{d(x+z)}{dt} = x + z \quad \text{and} \quad \frac{d(x-z)}{dt} = -(x-z),$$

  which yields

  $$x + z = c_1(s)e^t \quad \text{and} \quad x - z = c_2(s)e^{-t}.$$

  Using the initial conditions, we evaluate c_1 and c_2, then solve for x and z:

  $$x = \frac{3}{2}se^t - \frac{1}{2}se^{-t}, \quad y = e^t, \quad z = \frac{3}{2}se^t + \frac{1}{2}se^{-t}.$$

  Notice that z is defined for all s and t, but if we eliminate s and t in favor of x and y we obtain our solution

  $$u(x,y) = x\frac{3y^2+1}{3y^2-1},$$

  which exists for $|y| < \frac{1}{\sqrt{3}}$: a blow-up singularity has developed at $y = \frac{1}{\sqrt{3}}$, which is where $x_s y_t - y_s x_t$ vanishes.

- **Semilinear Equations.** Consider the semilinear equation

  $$a(x,y)u_x + b(x,y)u_y = c(x,y,u), \tag{18}$$

  with initial curve $(f(s), g(s), h(s))$. The characteristic equations become

  $$\frac{dx}{dt} = a(x,y), \quad \frac{dy}{dt} = b(x,y), \quad \frac{dz}{dt} = c(x,y,z)$$

  with the initial conditions

  $$x = f(s), \quad y = g(s), \quad z = h(s), \quad s \in \mathbb{R}.$$

  The first two equations form a system, which may be solved to obtain a curve $(x(t), y(t))$ in the xy-plane, called **projected characteristics**. If we first find the projected characteristics, we can then integrate the third characteristic equation to find z.

⊙ Moreover, regarding the problem of solving for s and t in terms of x and y, the inverse function theorem tells us that this can be achieved provided the Jacobian matrix is nonsingular

$$J \equiv \det \begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix} = x_s y_t - x_t y_s \neq 0. \tag{19}$$

Notice that **this condition is independent of the behavior of z.**

– In particular, at $t = 0$ we obtain the condition

$$f'(s)b(f(s), g(s)) - g'(s)a(f(s), g(s)) \neq 0, \tag{20}$$

which geometrically means that the projection of Γ into the xy-plane is a curve $\gamma = (f(s), g(s))$ that is nowhere parallel to the vector field (a, b).

– (18) implies by continuity that (19) holds at least for small values of t so we have the following result.

Proposition. *If the initial curve $\gamma = (f(s), g(s))$ satisfies (20), then there exists a unique solution $u(x, y)$ of (18) in a neighborhood of γ.*

However, away from γ (i.e. for large values of t) the solution may develop a blow up type of singularity where $J = 0$;
in fact, even if (19) holds for **all** values of s and t, the solution may develop a **blow up type** of singularity if the equation for $\frac{dz}{dt}$ is nonlinear.

Example 3. Consider the initial value problem

$$u_x + 2u_y = u^2$$

with

$$u(x, 0) = h(x), \quad x \in \mathbb{R}.$$

The characteristics are solutions of the system

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2, \quad \frac{dz}{dt} = z^2$$

and the initial curve Γ_0 has the parametrization

$$x = f(s) = s, \quad y = g(s) = 0, \quad z = h(s), \quad s \in \mathbb{R}.$$

We may integrate the first two equations (treating s as a constant) to find

$$x(s, t) = t + c_1(s), \quad y(s, t) = 2t + c_2(s),$$

where the functions $c_1(s)$ and $c_2(s)$ may be determined from the initial conditions:

$$x(s, 0) = c_1(s) = s, \quad y(s, 0) = c_2(s) = 0,$$

so that

$$x = t + s, \quad y = 2t.$$

Notice that (19) holds for all s and t, and we can explicitly solve for s and t to find

$$s = x - \frac{y}{2}, \quad t = \frac{y}{2}.$$

We may integrate the equation for z to find $z(s,t) = -\frac{1}{t+c_3(s)}$ and use the initial condition $z(s,0) = h(s)$ to evaluate $c_3 = -1/h(s)$, so

$$z(s,t) = -\frac{h(s)}{1 - th(s)}.$$

Finally, we may eliminate s and t to express our solution as

$$u(x,y) = \frac{h(x - \frac{y}{2})}{1 - \frac{y}{2}h(x - \frac{y}{2})}.$$

– Notice that $u(x,0) = h(x)$ and the solution $u(x,y)$ is certainly well defined for small enough values of y (assuming h is bounded);
however, u may become infinite if y becomes large enough to cause the denominator to vanish.
– Even though the equations for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are linear and the solutions exist for all s and t, the equations for $\frac{dz}{dt}$ is **nonlinear** and may produce a singularity.

Example 4. Consider the PDE

$$u_x + xu_y = u^2. \tag{21}$$

We find the projected characteristics by solving

$$\frac{dx}{1} = \frac{dy}{x}$$

to find the parabolas

$$y = \frac{1}{2}x^2 + C, \quad C\text{: constant.}$$

Take γ to be the parabola $y = \frac{x^2}{2}$. Then for $\Gamma = (\gamma, z(\gamma))$ to be characteristic, we must have

$$\frac{dz}{z^2} = dx.$$

We may integrate this equation to find

$$-\frac{1}{z} = x + c, \quad c\text{: constant.}$$

The constant c is determined by picking a point over γ for Γ to pass through.
– For example, if Γ passes through $(0,0,z_0)$, then $c = -\frac{1}{z_0}$, and Γ is given by

$$z = -\frac{z_0}{1 - z_0 x}.$$

For an arbitrary C^1 function f,

$$y - \frac{x^2}{2} = f(x + \frac{1}{z})$$

is a solution of (21).
– Along Γ, $y - \frac{x^2}{2} = 0$ and $x + \frac{1}{z} = \frac{1}{z_0}$, so that the solution will pass through Γ provided $f(\frac{1}{z_0}) = 0$; with this sole restriction on f we see that there is an infinite number of solutions passing through Γ.

Remark. An interesting situation occurs when we want to find a solution of the equation (18) in a smooth domain $\Omega \subset \mathbb{R}^2$, which assumes prescribed values on a subset of the boundary $\gamma = \partial\Omega$.

– For the solvability of the problem, we have to assign the Cauchy data only on the so called **inflow boundary** γ_i defined by

$$\gamma_i = \{\sigma \in \gamma : \mathbf{w} \cdot \nu < 0\}$$

where ν is the unit outward normal to γ.

– If a smooth Cauchy data is prescribed on γ_i, a smooth solution is obtained by defining

(i) u to be constant along each projected characteristic which meets the inflow boundary exactly once,

(ii) u to be piecewise constant on those projected characteristics which is tangent to the inflow boundary at some point.

Observe that the points at which $\mathbf{w}$ is tangent to $\mathbf{w}$ is characteristic.

Higher Dimensions

We now replace (1) with the equation

$$\sum_{i=1}^{n} a_i(x_1, \cdots, x_n, u) u_{x_i} = c(x_1, \cdots, x_n, u). \tag{22}$$

The **characteristic curves** are now the integral curves of the system of $n+1$ equations of $n+1$ unknowns

$$\frac{dx_i}{dt} = a_i(x_1, \cdots, x_n, z), \quad \frac{dz}{dt} = c(x_1, \cdots, x_n, z).$$

As in the case $n = 2$, it is clear that if u is a solution of (1), then the hypersurface

$$M_u = \{(x_1, \cdots, x_n, u(x_1, \cdots, x_n))\} \subset \mathbb{R}^{n+1}$$

is a union of characteristic curves.

– Now suppose we are given a bijection

$$\sigma = (\sigma_1, \cdots, \sigma_n) : \mathcal{D} \to \mathbb{R}^n, \qquad f_i = f_i(s_1, \cdots, s_{n-1})$$

where $\mathcal{D} \subset \mathbb{R}^{n-1}$ is a compact $(n-1)$-dimensional manifold-with-boundary, and a function $h : \mathcal{D} \to \mathbb{R}$, $h = h(s_1, \cdots, s_{n-1})$.

– We can produce a solution u of (22) with

$$u(\sigma(s)) = h(s), \quad \forall s \in \mathcal{D},$$

by taking the union of characteristic curves through all points $(\sigma(s), h(s)) \in \mathbb{R}^{n+1}$.

– Here we require that the matrix

$$\begin{pmatrix} D_1\sigma_1(s) & \cdots & D_{n-1}\sigma_1(s) & a_1(\sigma(s),h(s)) \\ \vdots & \ddots & \vdots & \vdots \\ D_1\sigma_n(s) & \cdots & D_{n-1}\sigma_n(s) & a_n(\sigma(s),h(s)) \end{pmatrix}$$

be non-singular. This means that

(1) the matrix $(D_j\sigma_i(s))$ must have rank $n-1$, so that σ is an imbedding and $\sigma(\mathcal{D}) \subset \mathbb{R}^n$ is a hypersurface, and
(2) the vector $(a_1(\sigma(s),h(s)), \cdots, a_n(\sigma(s),h(s)))$ must not lie in the tangent space of $\sigma(\mathcal{D})$.

Thus the method of characteristics generates an n-dimensional integral manifold parameterized by $(s_1, \cdots, s_{n-1}, t)$. The solution $u(x_1, \cdots, x_n)$ is obtained by solving for $(s_1, \cdots, s_{n-1}, t)$ in terms of the variables $(x_1, \cdots, x_n)$.

Example 5. Consider the initial value problem

$$u_{x_1} + x_1 u_{x_2} - u_{x_3} = u$$

with

$$u(x_1, x_2, 1) = x_1 + x_2, \quad x_1, x_2 \in \mathbb{R}.$$

The characteristics are solutions of the system

$$\frac{dx_1}{dt} = 1, \quad \frac{dx_2}{dt} = x_1, \quad \frac{dx_3}{dt} = -1, \quad \frac{dz}{dt} = z.$$

and the initial surface Γ_0 is the hyperplane $x_3 = 1$, $z = x_1 + x_2$, which is noncharacteristic and has the parametrization

$$x_1 = s_1, \quad x_2 = s_2, \quad x_3 = 1, \quad z = s_1 + s_2.$$

We find

$$x_1 = t + s_1, \quad y = \frac{1}{2}t^2 + s_1 t + s_2, \quad x_3 = -t + 1, \quad z = (s_1 + s_2)e^t.$$

We can then solve for s_1, s_2 and t and plug into x to find

$$u(x_1, x_2, x_3) = (x_1 + x_2 + (x_3 - 1)[1 + x_1 + \frac{1}{2}(x_3 - 1)])e^{1-x_3}.$$

Notice that the solution exists for all values of x_1, x_2 and x_3.