

## Geometrical Optics.

- In geometric optics, wave propagation is studied only in terms of the position of the wave front.
- If we have a light wave in the  $xy$ -plane, then its wave front at any time is a curve. Let us suppose that there is a function  $u(x, y)$  such that the position of the wave front at time  $t$  is given by the level curve  $u(x, y) = t$ .
- If  $c(x, y)$  denotes the **propagation speed** of a wave at  $(x, y)$ , then the curve  $(x(t), y(t))$  is a **light ray** if at each point its speed coincides with  $c(x, y)$  and its direction is orthogonal to the wave front

$$\begin{cases} \sqrt{(\dot{x})^2 + (\dot{y})^2} = c(x, y), \\ (\dot{x}, \dot{y}) \text{ and } \nabla u \text{ are collinear.} \end{cases}$$

- A point  $(x(t), y(t))$  on a ray satisfies the identity

$$(1) \quad u(x(t), y(t)) = t$$

and its velocity vector  $\mathbf{v} = (\dot{x}, \dot{y})$  is parallel to  $\nabla u$ . Therefore

$$\nabla u \cdot \mathbf{v} = |\nabla u| |\mathbf{v}| = c |\nabla u|.$$

On the other hand, differentiating (1) we obtain

$$\nabla u \cdot \mathbf{v} = u_x \dot{x} + u_y \dot{y} = 1,$$

from which we obtain  $c|\nabla u| = 1$ , i.e.

$$(2) \quad c^2(u_x^2 + u_y^2) = 1, \quad (c > 0).$$

Equation (2) is called the **eikonal equation** of geometrical optics in two-dimensions.

- Geometrically, if we fix a point  $(x_0, y_0, z_0)$ , the equation  $c^2(p^2 + q^2) = 1$  states that the family of planes

$$z - z_0 = p(x - x_0) + q(y - y_0),$$

tangent to an integral surface at  $(x_0, y_0, z_0)$ , all make a fixed angle

$$\theta = \arctan |\nabla u|^{-1} = \arctan c$$

with the  $z$  axis.

- This family envelopes the Monge cone which is the circular cone

$$(x - x_0)^2 + (y - y_0)^2 = c^2(z - z_0)^2.$$

Since the Monge cone is simply the union of all light rays passing through the point  $(x_0, y_0, z_0)$ . We call it the **light cone**, with opening angle  $2\theta$ .

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

- The eikonal equation is of the form (1) with

$$F(x, y, u, p, q) = \frac{1}{2}[c^2(p^2 + q^2) - 1].$$

The characteristic system is

$$(3) \quad \begin{cases} \frac{dx}{d\tau} = c^2 p, & \frac{dy}{d\tau} = c^2 q, & \frac{dz}{d\tau} = c^2 p^2 + c^2 q^2 = 1 \\ \frac{dp}{d\tau} = 0, & \frac{dq}{d\tau} = 0. \end{cases}$$

Notice that  $p$  and  $q$  are constant along the characteristics.

- ⊙ For initial values  $(x_0, y_0, z_0, p_0, q_0)$  with  $c^2(p_0^2 + q_0^2) = 1$ , the characteristic is the straight line

$$x = c^2 p_0 t + x_0, \quad y = c^2 q_0 t + y_0, \quad z = t + z_0,$$

which is a light ray.

- ⊙ For the Cauchy problem, suppose the initial curve  $\Gamma$  is given parametrically by

$$x = f(s), \quad y = g(s), \quad z = h(s).$$

We must complete  $\Gamma$  into an initial strip by solving for  $\phi$  and  $\psi$  the system

$$(4) \quad \begin{cases} c^2(\phi(s)^2 + \psi(s)^2) = 1 \\ \phi(s)f'(s) + \psi(s)g'(s) = h'(s). \end{cases}$$

This system has **two real distinct solutions** if

$$(5) \quad f'(s)^2 + g'(s)^2 > c^2 h'(s)^2,$$

while it has **no real solution** if

$$(6) \quad f'(s)^2 + g'(s)^2 < c^2 h'(s)^2.$$

- (i) If (5) holds,  $\Gamma_0$  forms an angle greater than  $\theta$  with the  $z$ -axis and therefore it is exterior to the light cone.
  - In this case, we say that  $\Gamma_0$  is **space-like** and we can find two different solutions of the Cauchy problem.
- (ii) If (6) holds,  $\Gamma_0$  forms an angle less than  $\theta$  with the  $z$ -axis and therefore it is contained in the light cone.
  - In this case, we say that  $\Gamma_0$  is **time-like** and the Cauchy problem does not have any solution.
- Given a space-like curve  $\Gamma_0$  and  $\phi, \psi$  solutions of the system (4), the corresponding characteristic strip is, for  $s$  fixed,

$$\begin{aligned} x(s) &= f(s) + c^2 \phi(s)t, & y(s) &= g(s) + c^2 \psi(s)t, & z(s) &= h(s) + t \\ p(s) &= \phi(s), & q(s) &= \psi(s). \end{aligned}$$

Observe that the point  $(x(t), y(t))$  moves along the characteristic with speed  $\sqrt{\dot{x}^2(t) + \dot{y}^2(t)} = \sqrt{\phi^2(s) + \psi^2(s)} = c$  and direction  $(\phi(s), \psi(s)) = (p(t), q(t))$ .

- Therefore, the characteristic lines are coincident with the light rays.
- Moreover, we see that the fronts  $\gamma_t$  can be constructed from  $\gamma_0$  by shifting any point on  $\gamma_0$  along a ray at a distance  $ct$ .

Thus, the wave fronts constitute a family of “parallel” curves.

**Example.** Let us solve (18) with initial conditions

$$\Gamma : x = \cos s, \quad y = \sin s, \quad z = 0.$$

To complete  $\Gamma$  to a strip, we must find  $\phi$  and  $\psi$  satisfying

$$\begin{cases} c^2(\phi^2(s) + \psi^2(s)) = 1, \\ -\phi(s) \sin s + \psi(s) \cos s = 0. \end{cases}$$

We find the two solutions

$$(\phi(s), \psi(s)) = \pm \frac{1}{c}(\cos s, \sin s)$$

which lead to the values

$$x = \cos s(1 \pm ct) \quad y = \sin s(1 \pm ct) \quad z = t.$$

In other words, we can write

$$(7) \quad x^2 + y^2 = (1 \pm ct)^2.$$

These two solutions of the Cauchy problem correspond to two distinct physical situations: Taking the positive sign in (7) produces a wave front moving away from the origin, and taking the negative sign produces a wave front which will converge on the origin at time  $c^{-1}$ .

- We may also study solutions of (1) by means of the **complete integrals** and **envelopes**.
- Consider the family of solutions

$$(8) \quad u(x, y; a, b) = c^{-1}(x \cos a + y \sin a) + b,$$

which are all planes making an angle  $\theta = \arctan c$  with the  $z$ -axis; at any fixed time  $z = t$ , the wave front is a straight line.

- This family (8) is a complete integrable, even though

$$D = \det \begin{pmatrix} u_{xa} & u_{ya} \\ u_{xb} & u_{yb} \end{pmatrix} \equiv 0,$$

but  $(a, b) \mapsto (u, u_x)$  is invertible away from  $a = \pm n\pi$ , and  $(a, b) \mapsto (u, u_y)$  is invertible near  $a = \pm n\pi$ .

- ⊙ We can take envelope of (8) to generate additional solutions of (1).

- For example, with  $b \equiv 0$ , the family

$$u(x, y; a) = c^{-1}(x \cos a + y \sin a)$$

are planes through the origin  $(x, y) = (0, 0)$  corresponding to the possible tangent planes to solutions.

- If we take the envelope of these solutions, we obtain

$$a = \arctan \frac{y}{x};$$

this produces the solution

$$u(x, y) = c^{-1} \sqrt{x^2 + y^2},$$

which coincides with the light cone, as should be expected from its identification with the Monge cone.

- We can use the light cone solutions to generate additional solutions as envelopes.
  - For example, suppose we want to find the solutions of (1) passing through a curve  $\gamma$  in the  $xy$ -plane.
  - Parameterize  $\gamma$  by  $(f(s), g(s))$  and let  $C_s$  denote the corresponding family of light cones along  $\gamma$ .
  - If we form an envelope of these light cones  $C_s$ , we shall get a solution of (1) passing through  $\gamma$ ; for small  $t$  the wave front will be a curve  $\gamma_t$  that is everywhere at a distance  $ct$  from  $\gamma$ .
  - This process realizes the wave front as the envelope of circular wave fronts; this principle, attributed to Huygens, is in fact foundational to the theory of geometric optics.