**Hopf-Lax Formula**

**Lemma 1.** *Suppose that for each* $\mathbf{x} \in \mathbb{R}^n$,

$$\lim_{\mathbf{q}\to\infty} \frac{L(\mathbf{x},\mathbf{q})}{|\mathbf{q}|} = \infty. \tag{1}$$

*Then*

$$\sup_{\mathbf{q}\in\mathbb{R}^n} \{\mathbf{p}\cdot\mathbf{q} - L(\mathbf{x},\mathbf{q})\} < \infty \tag{2}$$

*and* $\exists \mathbf{q}^* \in \mathbb{R}^n$ *such that*

$$\sup_{\mathbf{q}\in\mathbb{R}^n} \{\mathbf{p}\cdot\mathbf{q} - L(\mathbf{x},\mathbf{q})\} = \mathbf{p}\cdot\mathbf{q}^* - L(\mathbf{x},\mathbf{q}^*).$$

*Proof.* Fixing $\mathbf{p}$, by (1), there exists a constant $A$ such that

$$L(\mathbf{x},\mathbf{q}) \geq (|\mathbf{p}|+1)|\mathbf{q}|, \quad \text{if } |\mathbf{q}| \geq A.$$

Consequently, if $|\mathbf{q}| \geq A$,

$$\mathbf{p}\cdot\mathbf{q} - L(\mathbf{x},\mathbf{q}) \leq -|\mathbf{q}|$$

and hence

$$\mathbf{p}\cdot\mathbf{q} - L(\mathbf{x},\mathbf{q}) \leq -L(\mathbf{x},0) = \mathbf{p}\cdot\mathbf{q} - L(\mathbf{x},\mathbf{q})\Big|_{\mathbf{q}=\mathbf{0}},$$

if $|\mathbf{q}| \geq C \equiv \max(A, -L(\mathbf{x},0))$. Hence

$$\sup_{\mathbf{q}\in\mathbb{R}^n} \{\mathbf{p}\cdot\mathbf{q} - L(\mathbf{x},\mathbf{q})\} = \max_{\mathbf{q}\in B_C(\mathbf{0})} \{\mathbf{p}\cdot\mathbf{q} - L(\mathbf{x},\mathbf{q})\}.$$

The continuity of $L$ then yields the existence of $\mathbf{q}^*$. $\square$

**Lemma 2.** *Suppose that* $L \in C^1$ *such that for each* $\mathbf{x} \in \mathbb{R}^n$ *(1) is satisfied and*

$$\mathbf{q} \mapsto L(\mathbf{x},\mathbf{q}) \quad \textit{is convex}$$

*then the Legendre transformation of* $L$ *is*

$$L^*(\mathbf{x},\mathbf{p}) := \sup_{\mathbf{q}\in\mathbb{R}^n} \{\mathbf{p}\cdot\mathbf{q} - L(\mathbf{x},\mathbf{q})\} \quad (\mathbf{p} \in \mathbb{R}^n).$$

*Proof.* Suppose $\mathbf{q}^*$ ia a maximum point of the mapping $\mathbf{q} \mapsto \mathbf{p}\cdot\mathbf{q} - L(\mathbf{x},\mathbf{q}^*)$ provided by Lemma 1. Then $\mathbf{q}^*$ being a critical point of this mapping, we have

$$\mathbf{p} = D_{\mathbf{q}} L(\mathbf{x},\mathbf{q}^*),$$

and

$$\mathbf{q}^* = \mathbf{q}(\mathbf{x},\mathbf{p}).$$

Thus

$$\sup_{\mathbf{q}\in\mathbb{R}^n} \{\mathbf{p}\cdot\mathbf{q} - L(\mathbf{x},\mathbf{q})\} = D_{\mathbf{q}} L(\mathbf{x},\mathbf{q}(\mathbf{x},\mathbf{p})) \cdot -L(\mathbf{x},\mathbf{q}(\mathbf{x},\mathbf{p}));$$

the right hand side is exactly the definition of the Legendre transformation of $L$. $\square$

- Thus, if $L$ is convex in $\mathbf{q}$ and satisfies (1), we define the Legendre transform of $L$ to be $L^*(\mathbf{x},\mathbf{p}) = \sup_{\mathbf{q}\in\mathbb{R}^n} \{\mathbf{p}\cdot\mathbf{q} - L(\mathbf{x},\mathbf{q})\}$ and the Hamiltonian associated with $L$ is

  $$H(\mathbf{x},\mathbf{p}) = L^*(\mathbf{x},\mathbf{p}) = \sup_{\mathbf{q}\in\mathbb{R}^n} \{\mathbf{p}\cdot\mathbf{q} - L(\mathbf{x},\mathbf{q})\}. \tag{3}$$

  This tells us how, under the convexity assumption and (1), to obtain $H$ from $L$.
- Now we ask the converse question: "**given** $H$**, how do we obtain** $L$?"

**Theorem.** *Suppose $L$ is convex, satisfies (1) and $H$ is defined by (3). Then for each $\mathbf{x} \in \mathbb{R}^n$,*

*(i) $\mathbf{p} \mapsto H(\mathbf{x}, \mathbf{p})$ is convex, with*

$$\lim_{|\mathbf{p}|\to\infty} \frac{H(\mathbf{x}, \mathbf{p})}{|\mathbf{p}|} = \infty.$$

*(ii) Furthermore,*

$$L(\mathbf{x}, \mathbf{q}) = H^*(\mathbf{x}, \mathbf{q}) = \sup_{\mathbf{p}\in\mathbb{R}^n} \{\mathbf{p} \cdot \mathbf{q} - H(\mathbf{x}, \mathbf{p})\}. \tag{4}$$

*Proof.* (i) For each fixed $\mathbf{x}$ and $\mathbf{q}$, the function

$$\mathbf{p} \mapsto \mathbf{p} \cdot \mathbf{q} - L(\mathbf{x}, \mathbf{q})$$

is linear, and so the mapping

$$\mathbf{p} \mapsto H(\mathbf{x}, \mathbf{p}) = L^*(\mathbf{x}, \mathbf{p}) = \sup_{\mathbf{q}\in\mathbb{R}^n} \{\mathbf{p} \cdot \mathbf{q} - L(\mathbf{x}, \mathbf{q})\}$$

is convex. Moreover, fix $\lambda > 0$. Then

$$\begin{aligned} H(\mathbf{x}, \mathbf{p}) &= \sup_{\mathbf{q}\in\mathbb{R}^n} \{\mathbf{p} \cdot \mathbf{q} - L(\mathbf{x}, \mathbf{q})\} \\ &\geq \lambda|\mathbf{p}| - L(\mathbf{x}, \lambda\frac{\mathbf{p}}{|\mathbf{p}|}), \qquad \text{choosing } \mathbf{q} = \lambda\frac{\mathbf{p}}{|\mathbf{p}|}, \\ &\geq \lambda|\mathbf{p}| - \max_{\mathbf{q}\in B_\lambda(\mathbf{0})} L(\mathbf{x}, \mathbf{q}). \end{aligned}$$

By the continuity of $L$, $\max_{\mathbf{q}\in B_\lambda(\mathbf{0})} L(\mathbf{x}, \mathbf{q})$ is bounded. Hence, we obtain

$$\liminf_{|\mathbf{p}|\to\infty} \frac{H(\mathbf{x}, \mathbf{p})}{|\mathbf{p}|} \geq \lambda, \quad \forall \lambda > 0.$$

**(ii)** In view of (3),

$$H(\mathbf{x}, \mathbf{p}) + L(\mathbf{x}, \mathbf{q}) \geq \mathbf{p} \cdot \mathbf{q}, \quad \forall \mathbf{p}, \mathbf{q} \in \mathbb{R}^n,$$

and consequently

$$L(\mathbf{x}, \mathbf{q}) \geq \sup_{\mathbf{p}\in\mathbb{R}^n} \{\mathbf{p} \cdot \mathbf{q} - H(\mathbf{x}, \mathbf{p})\} = H^*(\mathbf{x}, \mathbf{q}).$$

On the other hand

$$\begin{aligned} H^*(\mathbf{x}, \mathbf{q}) = L^{**}(\mathbf{x}, \mathbf{p}) &= \sup_{\mathbf{p}\in\mathbb{R}^n} \{\mathbf{p} \cdot \mathbf{q} - L^*(\mathbf{x}, \mathbf{p})\} \\ &= \sup_{\mathbf{p}\in\mathbb{R}^n} \{\mathbf{p} \cdot \mathbf{q} - \sup_{\mathbf{r}\in\mathbb{R}^n} \{\mathbf{p} \cdot \mathbf{r} - L(\mathbf{x}, \mathbf{r})\}\} \\ &= \sup_{\mathbf{p}\in\mathbb{R}^n} \inf_{\mathbf{r}\in\mathbb{R}^n} \{\mathbf{p} \cdot (\mathbf{q} - \mathbf{r}) + L(\mathbf{x}, \mathbf{r})\} \qquad (5) \\ &= \inf_{\mathbf{r}\in\mathbb{R}^n} \sup_{\mathbf{p}\in\mathbb{R}^n} \{\mathbf{p} \cdot (\mathbf{q} - \mathbf{r}) + L(\mathbf{x}, \mathbf{r})\} \end{aligned}$$

Now since $\mathbf{q} \mapsto L(\mathbf{x}, \mathbf{q})$ is convex, there exists $\mathbf{s} \in \mathbb{R}^n$ such that

$$L(\mathbf{x}, \mathbf{r}) \geq L(\mathbf{x}, \mathbf{q}) + \mathbf{s} \cdot (\mathbf{r} - \mathbf{q}) \quad \forall \mathbf{r} \in \mathbb{R}^n;$$

(if $L$ is differentiable at $\mathbf{q}$, take $\mathbf{s} = D_{\mathbf{q}} L(\mathbf{x}, \mathbf{q})$.) Taking $\mathbf{p} = \mathbf{s}$ in (5), we compute

$$H^*(\mathbf{x}, \mathbf{q}) \geq \inf_{\mathbf{r}\in\mathbb{R}^n} \{\mathbf{s} \cdot (\mathbf{q} - \mathbf{r}) + L(\mathbf{x}, \mathbf{r})\} \geq L(\mathbf{x}, \mathbf{q}). \qquad \square$$

**Hopf's formula**

- Return to the Hamilton-Jacobi equation $u_t + H(\mathbf{x}, Du) = 0$.
  Suppose $H$ satisfies the following conditions for each $\mathbf{x} \in \mathbb{R}^n$:
  (i) $\mathbf{p} \mapsto H(\mathbf{x}, \mathbf{p})$ is convex, and (ii) $\lim_{|\mathbf{p}|\to\infty} \frac{H(\mathbf{x},\mathbf{p})}{|\mathbf{p}|} = \infty$.
  According to Theorem 1, $L(\mathbf{x}, \mathbf{q}) = H^*(\mathbf{x}, \mathbf{q}) = \sup_{\mathbf{p}\in\mathbb{R}^n}\{\mathbf{p}\cdot\mathbf{q} - H(\mathbf{x}, \mathbf{p})\}$ satisfies similar hypotheses.

- Recall now that the calculus of variations problem with Lagrangian $L$ led to Hamilton's ODE for the associated Hamiltonian $H$.
- Since these ODE are in turn the characteristic equations of the Hamilton-Jacobi PDE, we are led to **conjecture** that **there is a direct connection between the PDE and the calculus of variations problem**.
- So if $\mathbf{x} \in \mathbb{R}^n$ and $t > 0$ be given, we should presumably try to minimize the action

$$\int_0^t L(\mathbf{w}(s), \dot{\mathbf{w}}(s))ds$$

  over functions $\mathbf{w} : [0, t] \to \mathbb{R}^n$ satisfying $\mathbf{w}(t) = \mathbf{x}$.
- But what should we take for $\mathbf{w}(0)$?
  As we must somehow take into account the initial condition for our PDE, let us try to modify the action to include the function $g$ evaluated at $\mathbf{w}(0)$:

$$\int_0^t L(\dot{\mathbf{w}}(s), \mathbf{w}(s))ds + g(\mathbf{w}(0)).$$

- We follow up on this by guessing that a solution of (1), (2) should be given by the formula

$$u(\mathbf{x}, t) \equiv \inf\{\int_0^t L(\mathbf{w}(s), \dot{\mathbf{w}}(s))ds + g(\mathbf{y})\Big|\mathbf{w}(0) = \mathbf{y}, \mathbf{w}(t) = \mathbf{x}\}, \tag{6}$$

  the infimum taken over all points $\mathbf{y} \in \mathbb{R}^n$ and all piecewise smooth functions $\mathbf{w}(s)$ with $\mathbf{w}(0) = \mathbf{y}$, $\mathbf{w}(t) = \mathbf{x}$.

- To simplify further, let us suppose that $H$, and thus $L$, do **not** depend on $\mathbf{x}$. We are therefore assuming that

$$H \ \text{ is convex and } \ \frac{H(\mathbf{p})}{|\mathbf{p}|} \to \infty \quad \text{as } |\mathbf{p}| \to \infty.. \tag{7}$$

  Then

$$L(\mathbf{q}) = H^*(\mathbf{q}) = \sup_{\mathbf{p}\in\mathbb{R}^n}\{\mathbf{p}\cdot\mathbf{q} - H(\mathbf{p})\}$$

  has the same properties. For this situation, formula (6) becomes

$$u(\mathbf{x}, t) \equiv \inf\{\int_0^t L(\dot{\mathbf{w}}(s))ds + g(\mathbf{y})\Big|\mathbf{w}(0) = \mathbf{y}, \mathbf{w}(t) = \mathbf{x}\}. \tag{8}$$

- We propose now to investigate **the sense in which $u$ so defined actually solves the Hamilton-Jacobi PDE**

$$\begin{cases} u_t + H(u_{\mathbf{x}}) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{9}$$

We henceforth assume also that

$$g : \mathbb{R}^n \to \mathbb{R} \quad \text{is Lipschitz;} \tag{10}$$

this means that

$$\operatorname{Lip}(g) := \sup_{\substack{\mathbf{x},\mathbf{y}\in\mathbb{R}^n \\ \mathbf{x}\neq\mathbf{y}.}} \left\{ \frac{g(\mathbf{x}) - g(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \right\} < \infty.$$

- First we note that formula (8) can be further simplified:

**Theorem.** *If $\mathbf{x} \in \mathbb{R}^n$ and $t > 0$, then*

$$u(\mathbf{x}, t) = \min_{\mathbf{y}\in\mathbb{R}^n} \left\{ tL\left(\frac{\mathbf{x} - \mathbf{y}}{t}\right) + g(\mathbf{y}) \right\}. \tag{11}$$

We call the expression on the right hand side of (10) the **Hopf-Lax formula**.

**Remark.** We can rewrite (11) as

$$u(\mathbf{x}, t) = \min_{\mathbf{y}\in\mathbb{R}^n} \max_{\mathbf{z}\in\mathbb{R}^n} \{\mathbf{z} \cdot (\mathbf{x} - \mathbf{y}) - tH(\mathbf{z}) + g(\mathbf{y})\}$$

Note that for each fixed $\mathbf{y}, \mathbf{z}$, the linear function

$$(\mathbf{x}, t) \mapsto \mathbf{z} \cdot (\mathbf{x} - \mathbf{y}) - tH(\mathbf{z}) + g(\mathbf{y})$$

solves the PDE in (9). Thus Hopf's formula builds a solution of (9) by taking appropriate "envelope" of such functions using minima and maxima.

*Proof.* **(i)** Fix any $\mathbf{y} \in \mathbb{R}^n$ and define

$$\mathbf{w}(s) := \mathbf{y} + \frac{s}{t}(\mathbf{x} - \mathbf{y}), \quad 0 \le s \le t.$$

Then

$$\int_0^t L(\dot{\mathbf{w}}(s))ds = tL\left(\frac{\mathbf{x} - \mathbf{y}}{t}\right).$$

The definition (8) of $u$ implies

$$u(\mathbf{x}, t) \le \int_0^t L(\dot{\mathbf{w}}(s))ds + g(\mathbf{y}) = tL\left(\frac{\mathbf{x} - \mathbf{y}}{t}\right) + g(\mathbf{y}),$$

and so

$$u(\mathbf{x}, t) \le \inf_{\mathbf{y}\in\mathbb{R}^n} \left\{ tL\left(\frac{\mathbf{x} - \mathbf{y}}{t}\right) + g(\mathbf{y}) \right\}.$$

**(ii)** On the other hand, if $\mathbf{w}(s)$ is any $C^1$ function satisfying $\mathbf{w}(t) = \mathbf{x}$, we have, **by Jensen's inequality**

$$L\left(\frac{1}{t}\int_0^t \dot{\mathbf{w}}(s)ds\right) \le \frac{1}{t}\int_0^t L(\dot{\mathbf{w}}(s))ds.$$

Thus if we write $\mathbf{y} = \mathbf{x}(0)$, we find

$$tL\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) + g(\mathbf{y}) \le \int_0^t L(\dot{\mathbf{w}}(s))ds + g(\mathbf{y}),$$

and hence

$$\inf_{\mathbf{y}\in\mathbb{R}^n}\left\{tL\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) + g(\mathbf{y})\right\} \le u(\mathbf{x},t).$$

**(iii)** We have so far shown that

$$u(\mathbf{x},t) = \inf_{\mathbf{y}\in\mathbb{R}^n}\left\{tL\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) + g(\mathbf{y})\right\}. \tag{12}$$

In order to show that the infimum is really a minimum, observe first that our setting $\mathbf{y} = \mathbf{x}$ in the expression on the right hand side of (12) gives the estimate

$$u(\mathbf{x},t) \le tL(\mathbf{0}) + g(\mathbf{x}). \tag{13}$$

Now since

$$\lim_{|\mathbf{q}|\to\infty}\frac{L(\mathbf{q})}{|\mathbf{q}|} = \infty$$

there exists a constant $A$ such that

$$L(\mathbf{q}) \ge 2(\mathrm{Lip}(g)+1)|\mathbf{q}| \quad \text{if } |\mathbf{q}| \ge A.$$

Consequently, if $|\mathbf{x}-\mathbf{y}| \ge tA$,

$$\begin{aligned} tL\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) + g(\mathbf{y}) &\ge 2(\mathrm{Lip}(g)+1)|\mathbf{x}-\mathbf{y}| + g(\mathbf{y}) \\ &\ge (\mathrm{Lip}(g)+2)|\mathbf{x}-\mathbf{y}| + g(\mathbf{x}) \\ &\ge (\mathrm{Lip}(g)+2)|\mathbf{x}-\mathbf{y}| - tL(\mathbf{0}) + u(\mathbf{x},t), \quad \text{by (13)}. \end{aligned}$$

Thus

$$tL\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) + g(\mathbf{y}) \ge u(\mathbf{x},t)$$

if $|\mathbf{x}-\mathbf{y}| \ge tB$, for

$$B \equiv \max\left[A, \frac{L(0)}{\mathrm{Lip}(g)+1}\right].$$

Hence (12) becomes

$$u(\mathbf{x},t) = \min_{\mathbf{y}\in B_{tB}(\mathbf{x})}\left\{tL\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) + g(\mathbf{y})\right\}. \quad \square$$

- We now commence a study of the various properties of the function $u$ defined by Hopf's formula.

**Lemma 3.** *The function $u$ satisfies*

$$|u(\mathbf{x},t)-u(\overline{\mathbf{x}},t)| \leq Lip(g)|\overline{\mathbf{x}}-\mathbf{x}|, \quad \forall t>0, \mathbf{x}, \overline{\mathbf{x}} \in \mathbb{R}^n, \tag{14}$$

*and*

$$u = g \quad \text{on } \mathbb{R}^n \times \{t=0\}.$$

*Proof.* **(i)** Fix $t>0$, $\mathbf{x}, \overline{\mathbf{x}} \in \mathbb{R}^n$. Choose $\mathbf{y} \in \mathbb{R}^n$ such that

$$tL\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) + g(\mathbf{y}) = u(\mathbf{x},t).$$

Then

$$\begin{aligned} u(\overline{\mathbf{x}},t)-u(\mathbf{x},t) =& \inf_{\mathbf{z}}\left\{tL\left(\frac{\overline{\mathbf{x}}-\mathbf{z}}{t}\right)+g(\mathbf{z})\right\} - tL\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) - g(\mathbf{y}) \\ \leq& g(\overline{\mathbf{x}}-\mathbf{x}+\mathbf{y}) - g(\mathbf{y}) \\ \leq& \mathrm{Lip}(g)|\overline{\mathbf{x}}-\mathbf{x}|. \end{aligned}$$

Interchanging the roles of $\overline{\mathbf{x}}$ and $\mathbf{x}$, we establish (14).

**(ii)** Now select $\mathbf{x} \in \mathbb{R}^n$, $t>0$. Setting $\mathbf{y}=\mathbf{x}$ in the expression on the right hand side of (12) gives the estimate

$$u(\mathbf{x},t) \leq tL(\mathbf{0}) + g(\mathbf{x}). \tag{15}$$

Furthermore

$$\begin{aligned} u(\mathbf{x},t)-g(\mathbf{x}) =& \min_{\mathbf{y}\in\mathbb{R}^n}\left\{tL\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right)+g(\mathbf{y})-g(\mathbf{x})\right\} \\ \geq& \min_{\mathbf{y}\in\mathbb{R}^n}\left\{-\mathrm{Lip}(g)|\mathbf{x}-\mathbf{y}| + tL\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right)\right\} \\ =& -t\max_{\mathbf{z}\in\mathbb{R}^n}\{\mathrm{Lip}(g)|\mathbf{z}|-L(\mathbf{z})\}, \quad \text{setting } \mathbf{z}=\frac{\mathbf{x}-\mathbf{y}}{t} \\ =& -t\max_{\mathbf{w}\in B_{\mathrm{Lip}(g)}(\mathbf{0})}\max_{\mathbf{z}\in\mathbb{R}^n}\{\mathbf{w}\cdot\mathbf{z}-L(\mathbf{z})\} \\ =& -t\max_{\mathbf{w}\in B_{\mathrm{Lip}(g)}(\mathbf{0})} H(\mathbf{w}). \end{aligned}$$

This inequality and (15) imply $|u(\mathbf{x},t)-g(\mathbf{x})| \leq Ct$ for

$$C \equiv \max(L(\mathbf{0}), \max_{B_{\mathrm{Lip}(g)}(\mathbf{0})} H). \qquad \square \tag{16}$$

**Lemma 4.** *For each* $\mathbf{x} \in \mathbb{R}^n$ *and* $0 < s < t$,

$$u(\mathbf{x},t) = \min_{\mathbf{y}\in\mathbb{R}^n} \left\{ (t-s)L\left(\frac{\mathbf{x}-\mathbf{y}}{t-s}\right) + u(\mathbf{y},s) \right\}. \tag{17}$$

*Proof.* **(i) Claim:** $\forall \mathbf{y} \in \mathbb{R}^n$**,** $u(\mathbf{x},t) \le (t-s)L\left(\frac{\mathbf{x}-\mathbf{y}}{t-s}\right) + u(\mathbf{y},s)$.
Indeed, fix $t > 0$, $\mathbf{x} \in \mathbb{R}^n$. Choose arbitrarily $\mathbf{y} \in \mathbb{R}^n$. Then for some $\mathbf{z} \in \mathbb{R}^n$,

$$u(\mathbf{y},s) = sL\left(\frac{\mathbf{y}-\mathbf{z}}{s}\right) + g(\mathbf{z}).$$

$$\begin{aligned}\Rightarrow u(\mathbf{x},t) - u(\mathbf{y},s) \le& \left[tL\left(\frac{\mathbf{x}-\mathbf{z}}{t}\right) + g(\mathbf{z})\right] - \left[sL\left(\frac{\mathbf{y}-\mathbf{z}}{s}\right) + g(\mathbf{z})\right] \\ =& tL\left(\frac{\mathbf{x}-\mathbf{z}}{t}\right) - sL\left(\frac{\mathbf{y}-\mathbf{z}}{s}\right). \end{aligned} \tag{18}$$

Now since $L$ is convex and

$$\frac{\mathbf{x}-\mathbf{z}}{t} = \left(1-\frac{s}{t}\right)\frac{\mathbf{x}-\mathbf{y}}{t-s} + \frac{s}{t}\frac{\mathbf{y}-\mathbf{z}}{s},$$

we have

$$L\left(\frac{\mathbf{x}-\mathbf{z}}{t}\right) \le \left(1-\frac{s}{t}\right)L\left(\frac{\mathbf{x}-\mathbf{y}}{t-s}\right) + \frac{s}{t}L\left(\frac{\mathbf{y}-\mathbf{z}}{s}\right).$$

Substituting this into (18), we obtain

$$u(\mathbf{x},t) - u(\mathbf{y},s) \le (t-s)L\left(\frac{\mathbf{x}-\mathbf{y}}{t-s}\right).$$

Therefore

$$u(\mathbf{x},t) \le \min_{\mathbf{y}\in\mathbb{R}^n} \left\{ (t-s)L\left(\frac{\mathbf{x}-\mathbf{y}}{t-s}\right) + u(\mathbf{y},s) \right\}.$$

**(ii) Claim:** $\exists \mathbf{y} \in \mathbb{R}^n$**,** $u(\mathbf{x},t) \ge (t-s)L\left(\frac{\mathbf{x}-\mathbf{y}}{t-s}\right) + u(\mathbf{y},s)$.
For this, select $\mathbf{w} \in \mathbb{R}^n$ such that

$$u(\mathbf{x},t) = tL\left(\frac{\mathbf{x}-\mathbf{w}}{t}\right) + g(\mathbf{w}).$$

$$\begin{aligned}\Rightarrow u(\mathbf{x},t) - u(\mathbf{y},s) \ge& \left[tL\left(\frac{\mathbf{x}-\mathbf{w}}{t}\right) + g(\mathbf{z})\right] - \left[sL\left(\frac{\mathbf{y}-\mathbf{w}}{s}\right) + g(\mathbf{w})\right] \\ =& tL\left(\frac{\mathbf{x}-\mathbf{w}}{t}\right) - sL\left(\frac{\mathbf{y}-\mathbf{w}}{s}\right), \quad \forall \mathbf{y} \ \text{ and } \ \forall s < t. \end{aligned} \tag{19}$$

On the other hand, if we set $\mathbf{y} = \frac{s}{t}\mathbf{x} + \left(1-\frac{s}{t}\right)\mathbf{w}$, then $\frac{\mathbf{x}-\mathbf{y}}{t-s} = \frac{\mathbf{x}-\mathbf{w}}{t} = \frac{\mathbf{y}-\mathbf{w}}{s}$, and hence

$$tL\left(\frac{\mathbf{x}-\mathbf{w}}{s}\right) - sL\left(\frac{\mathbf{y}-\mathbf{w}}{s}\right) = (t-s)L\left(\frac{\mathbf{x}-\mathbf{y}}{t-s}\right).$$

Substituting this into (19), we obtain

$$\begin{aligned} u(\mathbf{x},t) \ge& tL\left(\frac{\mathbf{x}-\mathbf{w}}{t}\right) - sL\left(\frac{\mathbf{y}-\mathbf{w}}{s}\right) + u(\mathbf{y},s) \\ \ge& \min_{\mathbf{y}\in\mathbb{R}^n} \left\{ (t-s)L\left(\frac{\mathbf{x}-\mathbf{y}}{t-s}\right) + u(\mathbf{y},s) \right\}. \quad \square \end{aligned}$$

**Lemma 5.** *The function $u$ is Lipschitz on $\mathbb{R}^n \times [0, \infty)$.*

*Proof.* By (14),

$$\mathrm{Lip}(u(\cdot, \bar{t})) \leq \mathrm{Lip}(g). \tag{20}$$

Hence it only remains to **claim**:

$$\forall \mathbf{x} \in \mathbb{R}^n, 0 < \bar{t} < t, |u(\mathbf{x}, t) - u(\mathbf{x}, \bar{t})| \leq C|t - \bar{t}|, \quad \text{for some constant } C. \tag{21}$$

For this, select $\mathbf{x} \in \mathbb{R}^n$, $0 < \bar{t} < t$. Setting $\mathbf{y} = \mathbf{x}$ in the expression on the right hand side of (17) gives the estimate

$$u(\mathbf{x}, t) \leq tL(\mathbf{0}) + u(\mathbf{x}, \bar{t}).$$

On the other hand, by (14) Consequently, (17), (20) and calculations like those employed in the proof of Lemma 3 gives

$$\begin{aligned} u(\mathbf{x}, t) - u(\mathbf{x}, \bar{t}) &= \min_{\mathbf{y} \in \mathbb{R}^n} \left\{ tL\left(\frac{\mathbf{x} - \mathbf{y}}{t}\right) + u(\mathbf{y}, \bar{t}) - u(\mathbf{x}, \bar{t}) \right\} \\ &\geq \min_{\mathbf{y} \in \mathbb{R}^n} \left\{ -\mathrm{Lip}(g)|\mathbf{x} - \mathbf{y}| + tL\left(\frac{\mathbf{x} - \mathbf{y}}{t}\right) \right\} \\ &= -t \max_{\mathbf{w} \in B_{\mathrm{Lip}(g)}(\mathbf{0})} H(\mathbf{w}). \end{aligned}$$

Hence (21) holds for the constant $C$ defined by (16). $\square$

- Now **Rademacher's Theorem** asserts that a Lipschitz function is differentiable almost everywhere.
  Consequently in view of Lemma 4 our function $u$ defined by Hopf's formula (11) is differentiable a.e.
- The next theorem asserts that $u$ in fact solves the Hamilton-Jacobi equation a.e.

**Theorem 5.** *Suppose $\mathbf{x} \in \mathbb{R}^n$, $t > 0$, and $u$ is differentiable at $(\mathbf{x}, t)$. Then*

$$u_t(\mathbf{x}, t) + H(Du(\mathbf{x}, t)) = 0.$$

*Proof.* **(i) Claim:** $\forall \mathbf{q} \in \mathbb{R}^n$, $u_t(\mathbf{x}, t) + \mathbf{q} \cdot Du(\mathbf{x}, t) - L(\mathbf{q}) \leq 0$.
Indeed, fix $\mathbf{q} \in \mathbb{R}^n$, $h > 0$. Then by Lemma 4,

$$\begin{aligned} u(\mathbf{x} + h\mathbf{q}, t + h) &= \inf_{\mathbf{y} \in \mathbb{R}^n} \left\{ hL\left(\frac{\mathbf{x} + h\mathbf{q} - \mathbf{y}}{h}\right) + u(\mathbf{y}, t) \right\} \\ &\leq hL(\mathbf{q}) + u(\mathbf{x}, t). \end{aligned}$$

Hence

$$\frac{u(\mathbf{x} + h\mathbf{q}, t + h) - u(\mathbf{x}, t)}{h} \leq L(\mathbf{q}).$$

Letting $h \to 0^+$, we obtain

$$u_t(\mathbf{x}, t) + \mathbf{q} \cdot Du(\mathbf{x}, t) \le L(\mathbf{q}).$$

This inequality is valid for all $\mathbf{q} \in \mathbb{R}^n$ and so

$$u_t(\mathbf{x}, t) + H(Du(\mathbf{x}, t)) = u_t(\mathbf{x}, t) + \sup_{\mathbf{q} \in \mathbb{R}^n} \{\mathbf{q} \cdot Du(\mathbf{x}, t) - L(\mathbf{q})\} \le 0.$$

**(ii) Claim:** $\exists \mathbf{q} \in \mathbb{R}^n$ **s.t.** $u_t(\mathbf{x}, t) + \mathbf{q} \cdot Du(\mathbf{x}, t) - L(\mathbf{q})\} \ge 0.$
For this, select $\mathbf{z} \in \mathbb{R}^n$ such that

$$u(\mathbf{x}, t) = tL\left(\frac{\mathbf{x} - \mathbf{z}}{t}\right) + g(\mathbf{z}),$$

and for $h > 0$, set

$$s = t - h, \quad \text{and} \quad \mathbf{y} = \frac{s}{t}\mathbf{x} + \big(1 - \frac{s}{t}\big)\mathbf{z}.$$

Then

$$\frac{\mathbf{x} - \mathbf{y}}{t - s} = \frac{\mathbf{x} - \mathbf{z}}{t} = \frac{\mathbf{y} - \mathbf{z}}{s}.$$

Consequently,

$$\begin{aligned} u(\mathbf{x}, t) - u(\mathbf{y}, s) \ge& tL\left(\frac{\mathbf{x} - \mathbf{z}}{t}\right) + g(\mathbf{z}) - \left[sL\left(\frac{\mathbf{y} - \mathbf{z}}{s}\right) + g(\mathbf{z})\right] \\ =& (t - s)L\left(\frac{\mathbf{x} - \mathbf{z}}{t}\right); \end{aligned}$$

that is,

$$\frac{u(\mathbf{x}, t) - u((\mathbf{x} - h(\frac{\mathbf{x} - \mathbf{z}}{t}), t - h)}{h} \ge L\left(\frac{\mathbf{x} - \mathbf{z}}{t}\right).$$

Letting $h \to 0^+$, we obtain

$$u_t(\mathbf{x}, t) + \frac{\mathbf{x} - \mathbf{z}}{t} \cdot Du(\mathbf{x}, t) \ge L\left(\frac{\mathbf{x} - \mathbf{z}}{t}\right).$$

Consequently

$$\begin{aligned} &u_t(\mathbf{x}, t) + H(Du(\mathbf{x}, t)) = u_t(\mathbf{x}, t) + \max_{\mathbf{q} \in \mathbb{R}^n} \{\mathbf{q} \cdot Du(\mathbf{x}, t) - L(\mathbf{q})\} \\ &\ge u_t(\mathbf{x}, t) + \frac{\mathbf{x} - \mathbf{z}}{t} \cdot Du(\mathbf{x}, t) - L\left(\frac{\mathbf{x} - \mathbf{z}}{t}\right) \ge 0. \qquad \square \end{aligned}$$

**Corollary 6.** *The function $u$ defined by Hopf's formula (11) is differentiable a.e. in $\mathbb{R}^n \times (0, \infty)$, and solves the initial value problem for Hamilton-Jacobi PDE*

$$\begin{cases} u_t + H(u_{\mathbf{x}}) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$