Sobolev's Inequality, Poincaré Inequality and Compactness

I. Sobolev inequality and Sobolev Embeddig Theorems

Theorem 1 (Sobolev's embedding theorem). *Given the bounded, open set* $\Omega \subset \mathbb{R}^n$ *with* $n \geq 3$ *and* $1 \leq p < n$, *then*

$$W_0^{1,p}(\Omega) \subset L^{\frac{np}{n-p}}(\Omega)$$

and $W_0^{1,p}(\Omega)$ *is continuously embedded in the space* $L^{\frac{np}{n-p}}(\Omega)$. *This means that the following estimate*

$$\|f\|_{L^{\frac{np}{n-p}}(\Omega)} \leq C\|Df\|_{L^p(\Omega)}, \quad \forall f \in W_0^{1,p}(\Omega). \tag{1}$$

holds true with a constant $C = C(n,p) \in (0,+\infty)$; *here we denote the weak gradient by* $Df = (D^{e_1}f, \cdots, D^{e_n}f) \in L^p(\Omega) \times \cdots \times L^p(\Omega)$.

Proof (L. Nirenberg). (i) It suffices to prove the inequality (1) for all $f \in C_0^\infty(\Omega)$. In this context we need the **generalized Hölder inequality**, namely, if $f_j \in L^{p_j}(\Omega)$, $j = 1, \cdots, m$, such that $p_1^{-1} + \cdots + p_m^{-1} = 1$, then there holds

$$\int_\Omega f_1(x)\cdots f_m(x)dx \leq \|f_1\|_{L^{p_1}(\Omega)} \cdots \|f_m\|_{L^{p_m}(\Omega)}, \tag{2}$$

which can be easily deduced from Hölder's inequality by induction.

(ii) At first, we deduce the estimate (1) in **the case** $p = 1$.
Noting that $f \in C_0^\infty(\Omega)$, we have the following representation for all $x \in \mathbb{R}^n$:

$$f(x) = \int_{-\infty}^{x_i} D^{e_i} f(x_1, \cdots, x_{i-1}, t, x_{i+1}, \cdots, x_n)dt.$$

This implies

$$|f(x)| \leq \int_{-\infty}^{x_i} |D^{e_i} f|dt \leq \int_{-\infty}^{\infty} |D^{e_i} f|dx_i,$$

and consequently

$$|f(x)|^{\frac{n}{n-1}} \leq \left(\prod_{i=1}^n \int_{-\infty}^{\infty} |D^{e_i} f|dx_i\right)^{\frac{1}{n-1}}.$$

We integrate this inequality succeesively with respect to the variables $x_1, \cdots, x_n$, using each time the generalized Hölder inequality with $p_1 = \cdots = p_m = n-1$ and $m = n-1$.

$$\begin{aligned}
&\int_{-\infty}^{\infty} |f(x)|^{\frac{n}{n-1}} dx_1 \\
&\quad\leq \left(\int_{-\infty}^{\infty} |D^{e_1} f|dx_i\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \left(\prod_{i=2}^n \int_{-\infty}^{\infty} |D^{e_i} f|dx_i\right)^{\frac{1}{n-1}} dx_1 \\
&\quad\leq \left(\int_{-\infty}^{\infty} |D^{e_1} f|dx_i\right)^{\frac{1}{n-1}} \prod_{i=2}^n \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |D^{e_i} f|dx_i dx_1\right)^{\frac{1}{n-1}}.
\end{aligned}$$

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A similar integration over the variables $x_2, \cdots, x_n$ yields

$$\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx \leq \left(\prod_{i=1}^{n} \int_{\mathbb{R}^n} |D^{e_i} f| dx_i \right)^{\frac{1}{n-1}},$$

and finally

$$\begin{aligned} \|f\|_{\frac{n}{n-1}} \leq & \left(\prod_{i=1}^{n} \int_{\mathbb{R}^n} |D^{e_i} f| dx_i \right)^{\frac{1}{n}} \\ \leq & \int_{\Omega} \left(\sum_{i=1}^{n} |D^{e_i} f| \right) dx \qquad (3) \\ \leq & \frac{1}{\sqrt{n}} \int_{\Omega} |Df| dx = \frac{1}{\sqrt{n}} \|Df\|_1, \quad \forall f \in C_0^{\infty}(\Omega). \end{aligned}$$

(iii) We now consider **the case** $1 < p < n$.

- Here we inser $|f|^{\gamma}$ with $\gamma > 1$ into (3) and obtain the following relation with the aid of Hölder's inequality and the condition $p^{-1} + q^{-1} = 1$:

$$\begin{aligned} \| |f|^{\gamma} \|_{\frac{n}{n-1}} \leq & \frac{1}{\sqrt{n}} \int_{\Omega} \Big| D|f|^{\gamma} \Big| dx \\ = & \frac{\gamma}{\sqrt{n}} \int_{\Omega} |f|^{\gamma-1} |Df| dx \\ \leq & \frac{\gamma}{\sqrt{n}} \| |f|^{\gamma-1} \|_q \|Df\|_p, \end{aligned}$$

and consequently

$$\| |f| \|^{\gamma}_{\gamma\frac{n}{n-1}} \leq \frac{\gamma}{\sqrt{n}} \| |f| \|^{\gamma-1}_{(\gamma-1)q} \|Df\|_p.$$

Choosing

$$\gamma = \frac{(n-1)p}{n-p} = \frac{np-p}{n-p},$$

we infer

$$\frac{\gamma n}{n-1} = (\gamma - 1)q = \frac{np}{n-p}.$$

Finally, we arrive at

$$\|f\|_{\frac{np}{n-p}} \leq \frac{\gamma}{\sqrt{n}} \|Df\|_p, \quad \forall f \in C_0^{\infty}(\Omega).$$

with the constant $C = \frac{np-n}{\sqrt{n}(n-p)}$.

(iv) If now $u \in W_0^{1,p}(\Omega)$, we approximate u in $W^{1,p}$-norm by C_0^{∞} functions u_m, and apply (1) to the difference $u_\ell - u_m$. It follows that $\{u_m\}$ is a Cauchy sequence in $L^{\frac{np}{n-p}}$. Thus u itself is contained in the same space and satisfies (1). $\square$

Corollary. *If $kp < n$, the Sobolev space $W_0^{k,p}(\Omega)$ is continuously embedded in $L^{\frac{np}{n-kp}}(\Omega)$. That is, there exists a number C depending only on k, p and n such that*

$$\|u\|_{L^{\frac{np}{n-p}}(\Omega)} \leq C\|u\|_{W_0^{k,p}(\Omega)}, \quad \forall f \in W_0^{1,p}(\Omega). \tag{4}$$

Proof. Suppose $kp < n$ and $u \in W_0^{k,p}(\Omega)$. Let $p^* = \frac{np}{n-p}$ Then, since $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leq k$, the Sobolev inequality implies

$$\|D^\beta u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}, \quad \text{if} \quad |\beta| \leq k-1,$$

and hence $u \in W^{k-1,p^*}(\Omega)$.

- Similarly, we find $u \in W^{k-2,p^{**}}(\Omega)$, where

  $$p^{**} = \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n},$$

  and

  $$\|D^\gamma u\|_{L^{p^{**}}(\Omega)} \leq C\|u\|_{W^{k,p^*}(\Omega)}, \quad \text{if} \quad |\gamma| \leq k-2,$$

- Proceeding thusly, we finally obtain, after k steps, that (4) holds and $u \in W^{0,q}(\Omega) = L^q(\Omega)$, for

  $$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

- Equipped with the extension operator E, we extend the embedding theorem from the Sobolev spaces $W_0^{k,p}(\Omega)$ to the spaces $W^{k,p}(\Omega)$, if Ω is a C^k-domain.
- Namely, if $u \in W^{k,p}(\Omega)$, we consider $Eu \in W_0^{k,p}(\Omega')$, for some domain Ω' containing Ω, which then is contained in $L^{\frac{np}{n-kp}}(\Omega')$, if $kp < n$.
  And hence $u \in L^{\frac{np}{n-kp}}(\Omega)$, by restriction from Ω' to Ω.
  Since $Eu = u$ on Ω and $\|Eu\|_{W^{k,p}(\Omega')} \leq c\|u\|_{W^{k,p}(\Omega)}$ depending on Ω, we have thus proved the following version of the Sobolev embedding theorem:

Sobolev Embedding Theorem*. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^k-domain. If $kp < n$, the Sobolev space $W^{k,p}(\Omega)$ is continuously embedded in $L^{\frac{np}{n-kp}}(\Omega)$. That is, there exists a number C depending only on k, p, n and Ω such that*

$$\|u\|_{L^{\frac{np}{n-p}}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega).$$

II. Poincaré Ineqiality.

Poincaré Inequalities. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. There exists a positive constant C_p such that, for every $u \in W_0^{k,p}(\Omega)$,*

$$\|u\|_{L^p(\Omega)} \leq C_P \|\nabla u\|_{L^p(\Omega)}. \tag{4}$$

Proof. First we prove the formula for $u \in C_0^\infty(\Omega)$; then, if $u \in W_0^{1,p}(\Omega)$, select a sequence $\{u_k\} \subset C_0^\infty(\Omega)$ converging to u in $W^{1,p}$-norm as $k \to \infty$, i.e.

$$\|v_k - v\|_{L^p(\Omega)} \to 0, \quad \|\nabla v_k - \nabla v\|_{L^p(\Omega)} \to 0$$

In particular

$$\|v_k\|_{L^p(\Omega)} \to \|v\|_{L^p(\Omega)}, \quad \|\nabla v_k\|_{L^p(\Omega)} \to \|\nabla v\|_{L^p(\Omega)}.$$

Since (4) holds for every v_k, we have

$$\|v_k\|_{L^p(\Omega)} \leq C_P \|\nabla v_k\|_{L^p(\Omega)}$$

Letting $k \to \infty$, we obtain (4) for u. Thus, it is enough to prove (4) for $u \in C_0^\infty(\Omega)$.

- To this purpose, from the divergence theorem, we may write

  $$\int_{\Omega \cap \{v>0\}} \operatorname{div}(v^p \mathbf{x})\, d\mathbf{x} = 0, \qquad \int_{\Omega \cap \{v<0\}} \operatorname{div}(v^p \mathbf{x})\, d\mathbf{x} = 0. \tag{5}$$

  since $v = 0$ on $\partial\Omega$. Now

  $$\operatorname{div}(v^p \mathbf{x}) = pv\nabla v \cdot \mathbf{x} + nv^p$$

  so that (5) yields

  $$\begin{aligned} \int_{\Omega \cap \{v>0\}} v^p\, d\mathbf{x} &= -\frac{p}{n} \int_{\Omega \cap \{v>0\}} v^{p-1} \nabla v \cdot \mathbf{x}\, d\mathbf{x}, \\ \int_{\Omega \cap \{v<0\}} v^p\, d\mathbf{x} &= -\frac{p}{n} \int_{\Omega \cap \{v<0\}} v^{p-1} \nabla v \cdot \mathbf{x}\, d\mathbf{x}. \end{aligned}$$

  Since Ω is bounded, we have $\max_{\mathbf{x} \in \Omega} |\mathbf{x}| = M < \infty$; therefore, using the Schwartz's inequality, we get

  $$\begin{aligned} \int_{\Omega \cap \{v>0\}} v^p\, d\mathbf{x} =& \left| \frac{p}{n} \int_{\Omega \cap \{v>0\}} v^{p-1} \nabla v \cdot \mathbf{x}\, d\mathbf{x} \right| \\ \leq& \frac{pM}{n} \left(\int_{\Omega \cap \{v>0\}} |v^{p-1}|^q\, d\mathbf{x} \right)^{1/q} \left(\int_{\Omega \cap \{v>0\}} |\nabla v|^p \right)^{1/p} \\ =& \frac{pM}{n} \left(\int_{\Omega \cap \{v>0\}} |v|^p\, d\mathbf{x} \right)^{1/q} \|\nabla v\|_{L^p(\Omega \cap \{v>0\})}. \end{aligned}$$

  Analogous estimate can be drived for the integral over $\Omega \cap \{v < 0\}$. From this it follows (4) with $C_P = pM/n$. $\square$

Inequality (4) implies that in $W_0^{1,p}(\Omega)$, the norm $\|u\|_{W^{1,p}}$ is equivalent to $\|\nabla\|_{L^p}$. Indeed,

$$\|u\|_{W^{1,p}} = (\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p)^{1/p},$$

and from (4)

$$\|\nabla u\|_{L^p} \leq \|u\|_{W^{1,p}} \leq (C_P^p + 1)^{1/p} \|\nabla u\|_{L^p}.$$

III. Compactness Theorem of Rellich and Kondrachov

We call the Banach space $(\mathfrak{B}_1, \|\cdot\|_1)$ is **compactly embedded** into the Banach space $(\mathfrak{B}_2, \|\cdot\|_2)$ if the injective mapping $I_1 : \mathfrak{B}_1 \to \mathfrak{B}_2$ is compact; this means that bounded sets in $\mathfrak{B}_1$ are mapped onto precompact sets in $\mathfrak{B}_2$.

Compactness Theorem of Rellich and Kondrachov.
Let Ω denote a bounded, domain in $\mathbb{R}^n$.

(i.1) *Let $1 \le p < n$. Then for all $1 \le q < \frac{np}{n-p}$, $W_0^{1,p}(\Omega)$ is compactly embedded into $L^q(\Omega)$. This means for each sequence $\{f_k\}_{k=1,2,\cdots} \subset W_0^{1,p}(\Omega)$ with $\|f\|_{W^{1,p}(\Omega)} \le s \in [0,\infty)$ we can select a subsequence $\{f_{k_\ell}\}_{\ell=1,2,\cdots}$ and element $f \in L^q(\Omega)$ satisfying $\lim_{\ell\to\infty} \|f_{k_\ell} - f\| = 0$.*

(i.2) *If Ω is Lipschize, then $W^{1,p}(\Omega)$ is compactly embedded into $L^q(\Omega)$.*

(ii.1) *$W_0^{1,2}(\Omega)$ is compactly embedded into $L^2(\Omega)$.*

(ii.2) *If Ω is Lipschize, then $W^{1,2}(\Omega)$ is compactly embedded into $L^q(\Omega)$.*

To prove (i.1) and (i.2), we shall use the following result.

Interpolation Inequality. *If the exponents $1 \le p \le q \le r$ fulfill*

$$\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r} \quad \text{with } \lambda \in [0,1],$$

then

$$\|f\|_q \le \|f\|_p^\lambda \|f\|_r^{1-\lambda}, \quad \forall f \in L^r(\Omega).$$

Proof of Interpolation Inequality. Noting

$$1 = \frac{\lambda q}{p} + \frac{(1-\lambda)q}{r} = \left(\frac{p}{\lambda q}\right)^{-1} + \left(\frac{r}{(1-\lambda)q}\right)^{-1}$$

we obtain

$$\begin{aligned}\|f\|_q =& \left(\int_\Omega |f|^{\lambda q}|f|^{(1-\lambda)q}dx\right)^{1/q} \\ \le& \left(\int_\Omega |f|^p dx\right)^{\frac{\lambda}{p}} \left(\int_\Omega |f|^r dx\right)^{\frac{1-\lambda}{r}} = \|f\|_p^\lambda \|f\|_r^{1-\lambda}. \qquad \square\end{aligned}$$

Proof of (i.1). Step 1. We start with an arbitrary sequence $\{f_k\}$ with

$$\|f\|_{W^{1,p}(\Omega)} \le s \in [0,\infty),$$

and make the transition to a sequence $\{g_k\}_{k=1,2,\cdots} \subset C_0^\infty(\Omega)$ with the property

$$\|g_k - f_k\|_{W^{1,p}(\Omega)} \le \frac{1}{k}.$$

The latter satisfies the restriction

$$\|g_k\|_{W^{1,p}(\Omega)} \le 1 + s, \quad \forall k \in \mathbb{N}.$$

If we manage to select a subsequence $\{g_{k_\ell}\}$ convergent in $L^1(\Omega)$ from the sequence $\{g_k\}$, then the sequence $\{f_{k_\ell}\}$ is convergent in $L^1(\Omega)$ as well; here we observe

$$\|g_k - f_k\|_{L^1(\Omega)} \le c\|g_k - f_k\|_{W^{1,p}(\Omega)} \le \frac{c}{k}.$$

Step 2. In order to show that the sequence $\{f_{k_\ell}\}$ converges even in the space $L^q(\Omega)$ with $1 < q < \frac{np}{n-p}$, we apply the interpolation inequality by choosing $\lambda \in (0,1)$ with the property

$$\frac{1}{q} = \lambda + (1-\lambda)\frac{n-p}{np}.$$

The interpolation inequality and the Sobolev inequality yield the estimate

$$\|f\|_q \le \|f\|_1^\lambda \|f\|_{np/(n-p)}^{1-\lambda} \le \|f\|_1^\lambda (C\|Df\|_p)^{1-\lambda}, \quad \forall f \in W_0^{1,p}(\Omega).$$

Therefore, we have

$$\|f_{k_\ell} - f_{k_m}\|_q \le \widetilde{C}\|f_{k_\ell} - f_{k_m}\|_1^\lambda \to 0, \quad \text{as } \ell, m \to \infty$$

Step 3. It still remains to select a subsequence in $L^1(\Omega)$ from the sequence

$$\{g_k\}_{k=1,2,\cdots} \subset C_0^\infty(\Omega).$$

Therefore, we take an arbitrary $\varepsilon \in (0,1)$ and consider the sequence of functions

$$g_{k,\varepsilon}(x) := \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \rho\left(\frac{x-y}{\varepsilon}\right) g_k(y)dy = \int_{\mathbb{R}^n} \rho(z) g_k(x - \varepsilon z) \in C_0^\infty(\Theta),$$

where

$$\Theta = \{x \in \mathbb{R}^n : \operatorname{dist}(x,\Omega) < 1\}.$$

For each fixed $\varepsilon \in (0,1]$, the sequence of functions $\{g_{k,\varepsilon}\}$ is **uniformly bounded** and **equicontinuous**, since we have the following estimates for all $x \in \Theta$:

$$|g_{k,\varepsilon}(x)| \le \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \rho\left(\frac{x-y}{\varepsilon}\right)|g_k(y)|dy \le \frac{C_0}{\varepsilon^n}\sup_{|\varepsilon|\le 1}\rho(x)$$

and

$$\begin{aligned} |Dg_{k,\varepsilon}(x)| \le& \frac{1}{\varepsilon^{n+1}}\int_{\mathbb{R}^n}\left|D\rho\left(\frac{x-y}{\varepsilon}\right)\right||g_k(y)|dy \\ \le& \varepsilon^{-(n+1)}\sup_{|z|\le 1}|D\rho(z)|\int_{\mathbb{R}^n}|g_k(y)|dy \\ \le& \frac{C_0}{\varepsilon^{n+1}}\sup_{|z|\le 1}|D\rho(z)|. \end{aligned}$$

Step 4. For each $\varepsilon > 0$, the Arzelá-Ascoli theorem thus yields a subsequence $\{g_{k_\ell,\varepsilon}\}$ of the sequence $\{g_{k,\varepsilon}\}$ converging uniformly in the set $\overline{\Omega}$.

- We now set $\varepsilon_m = \frac{1}{m}$ with $m \in \mathbb{N}$. Using the Cantor's diagonal procedure we select a subsequence $\{g_{k_\ell}\}$ of the sequence $\{g_k\}$ such that, for each fixed $m \in \mathbb{N}$, the sequence $\{g_{k_\ell,\varepsilon_m}\}$ converges uniformly in the set $\overline{\Omega}$.

Step 5. We have the inequality

$$\begin{aligned} |g_k(z) - g_{k,\varepsilon}(x)| &\leq \int_{|z|\leq 1} \rho(z)|g_k(x) - g_k(x-\varepsilon z)|dz \\ &\leq \int_{|z|\leq 1} \rho(z) \int_0^{\varepsilon} |Dg_k(x-tz)|dtdz, \end{aligned}$$

for all $x \in \Omega$, which implies the estimate

$$\int_{\Omega} |g_k(z) - g_{k,\varepsilon}(x)|dx \leq \varepsilon \int_{\Omega} |Dg_k(x-tz)|dx \leq C_1\varepsilon, \quad \forall k \in \mathbb{N}. \tag{5}$$

Choosing an arbitrary number $\varepsilon > 0$, we obtain the relation

$$\begin{aligned} &\|g_{k_{\ell_1}} - g_{k_{\ell_2}}\|_{L^1(\Omega)} \\ &\leq \|g_{k_{\ell_1}} - g_{k_{\ell_1,\varepsilon_m}}\|_{L^1(\Omega)} + \|g_{k_{\ell_1,\varepsilon_m}} - g_{k_{\ell_2,\varepsilon_m}}\|_{L^1(\Omega)} + \|g_{k_{\ell_2,\varepsilon_m}} - g_{k_{\ell_2}}\|_{L^1(\Omega)} \\ &\leq (2C_1 + |\Omega|)\varepsilon, \quad \forall \ell_1, \ell_2 \geq \text{some constant} \ \ \ell_0(\varepsilon). \end{aligned} \tag{6}$$

Consequently, $\{g_{k_\ell}\}$ represents a Cauchy sequence in $L^1(\Omega)$ and hence possesses a limit in $L^1(\Omega)$. $\square$

Proof of (ii.1). (a) We again start with a sequence $\{f_k\}$ with $\|f_k\|_{W^{1,2}(\Omega)} \leq s \in [0, \infty)$, and make the transition to a sequence $\{g_k\}_{k=1,2,\cdots} \subset C_0^{\infty}(\Omega)$ with the property $\|g_k - f_k\|_{W^{1,2}(\Omega)} \leq \frac{1}{k}$. Thus $\{g_k\}$ satisfies the restriction

$$\|g_k\|_{W^{1,2}(\Omega)} \leq 1 + s, \quad \forall k \in \mathbb{N}.$$

To prove Proposition 2, it suffices to select a subsequence $\{g_{k_\ell}\}$ convergent in $L^2(\Omega)$ from the sequence $\{g_k\}$.

(b) For this purpose, we again take an arbitrary $\varepsilon \in (0, 1)$ and consider the sequence of functions

$$g_{k,\varepsilon}(x) := \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \rho\left(\frac{x-y}{\varepsilon}\right) g_k(y)dy = \int_{\mathbb{R}^n} \rho(z) g_k(x - \varepsilon z) \in C_0^{\infty}(\Theta),$$

where

$$\Theta = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < 1\}.$$

As shown in **Step 3** of the proof of Proposition 1, the sequence of functions $\{g_{k,\varepsilon}\}$ is **uniformly bounded** and **equicontinuous**, for each fixed $\varepsilon \in (0, 1]$.

– Then, as in **Step 4** of the proof of Proposition 1, the Arzelá-Ascoli theorem and Cantor's digonal process yield a s subsequence $\{g_{k_\ell}\}$ of the sequence $\{g_k\}$ such that, for each fixed $m \in \mathbb{N}$, the sequence $\{g_{k_\ell,1/m}\}$ converges uniformly in the set $\overline{\Omega}$.

(c) To show that $\{g_{k_\ell}\}$ convergent in $L^2(\Omega)$, the crucial step is to establish, analogously to (5),

$$\|g_k(z) - g_{k,\varepsilon}(x)\|_{L^2(\Omega)} \leq \varepsilon \|Dg_k\|_{L^2(\Omega)}, \quad \forall k \in \mathbb{N}. \tag{7}$$

In fact, we have

$$
\begin{aligned}
|g_k(z) - g_{k,\varepsilon}(x)|^2 \leq & \left(\int_{|z|\leq 1} \rho(z)|g_k(x) - g_k(x-\varepsilon z)|dz \right)^2 \\
\leq & \left[\int_{|z|\leq 1} \rho(z) \left(\int_0^{\varepsilon} |Dg_k(x-tz)|dt \right) dz \right]^2,
\end{aligned}
$$

for all $x \in \Omega$, which implies the estimate

$$
\begin{aligned}
\int_\Omega |g_k(z) - g_{k,\varepsilon}(x)|^2 dx \leq & \int_\Omega \left[\int_{|z|\leq 1} \rho(z) \left(\int_0^{\varepsilon} |Dg_k(x-tz)|dt \right) dz \right]^2 dx \\
= & \int_\Omega \left[\int_{|z|\leq 1} (\rho(z))^{1/2} (\rho(z))^{1/2} \left(\int_0^{\varepsilon} |Dg_k(x-tz)|dt \right) dz \right]^2 dx \\
\leq & \left(\int_{|z|\leq 1} \rho(z) dz \right) \left(\int_{|z|\leq 1} \rho(z) \varepsilon^2 \int_\Omega |Dg_k(x)|^2 dx dz \right) \\
\leq & \varepsilon^2 \int_\Omega |Dg_k(x)| dx, \quad \forall k \in \mathbb{N}.
\end{aligned}
$$

Analogously to (6), we use the triangle inequality for L^2-norm and (7) to conclude that $\{g_{k_\ell}\}$ is a Cauchy sequence in $L^2(\Omega)$ and hence possesses a limit in $L^2(\Omega)$. □

Proof of (i.2) and (ii.2). If $1 < p < \infty$ and $\{f_k\}$ is a bounded sequence in $W^{1,p}(\Omega)$, we consider $Ef_k \in W_0^{1,p}(\Omega')$, for some domain Ω' containing Ω, i.e. $Eu = u$ on Ω and

$$
(8) \qquad \|Ef_k\|_{W^{1,p}(\Omega')} \leq c\|f_k\|_{W^{1,p}(\Omega)}, \quad \text{for some constant } c \text{ depending on } \Omega.
$$

By (8), the sequence $\{Ef_k\}$ is also bounded in $W^{1,p}(\Omega)$.

- Hence, if $p < n$, then (i) implies $\{Ef_k\}$ contains a subsequence $\{Ef_{k_\ell}\}$ which converges in $L^{np/(n-p)}(\Omega')$. Since $Ef_k = f_k$ in Ω, the sequence f_{k_ℓ} converges in $L^{np/(n-p)}(\Omega)$.
- By (ii.1), $\{Ef_k\}$ contains a subsequence $\{Ef_{k_\ell}\}$ which converges in $L^2(\Omega')$. Since $Ef_k = f_k$ in Ω, the sequence f_{k_ℓ} converges in $L^2(\Omega)$. □

IV. Poincaré Ineqiality: revisted.

- We have already proved that there exists a positive constant C_p such that, for every $u \in W_0^{k,p}(\Omega)$,

  (9) $$\|u\|_{L^p(\Omega)} \le C_P \|\nabla u\|_{L^p(\Omega)}.$$
- On the other hand, (9) cannot hold if $u =$constant.
- Roughly speaking, the hypotheses that guarantee the validity of (9) require that u **vanishes in some "nontrivial set"**. For instance, under each of the following conditions, (9) holds:

  (i) $u \in W^{1,p}_{0,\Gamma_0}(\Omega)$; i.e. u has zero trace on a nonempty relativly open subset $\Gamma_0 \subset \partial\Omega$;

  (ii) $u \in W^{1,2}(\Omega)$ and $u = 0$ on a set $E \subset \Gamma$ with positive measure $|E| = \alpha > 0$;

  (iii) $u \in W^{1,2}(\Omega)$ and $\int_\Omega u = 0$, i.e. u has mean value zero in Ω.

Poincaré Inequality: revisted. *Let Ω be a bounded, Lipschitz domain. Assume that u satisfies one of the hypotheses (i), (ii), (iii) above. Then, there exists C_P such that (8) holds.*

Proof. Assume that one of the hypotheses (i), (ii), (iii) holds. By contradiction suppose (9) is not true. This means that $\forall j \in \mathbb{N}$, $\exists u_j$ satisfies the same hypothesis such that

(10) $$\|u_j\|_{L^p(\Omega)} > j\|\nabla u_j\|_{L^p(\Omega)}.$$

Normalize u_j in $L^p(\Omega)$ by setting

$$w_j = \frac{u_j}{\|u_j\|_{L^p(\Omega)}}.$$

Then, from (10),

$$\|w_j\|_{L^p(\Omega)} = 1 \quad \text{and} \quad \|\nabla w_j\|_{L^p(\Omega)} < \frac{1}{j} \le 1.$$

Thus $\{w_j\}$ is bounded in $W^{1,p}(\Omega)$ and by Rellich's theorem there exists a sequence $\{w_{j_k}\}$ and w also satisfies the same hypothesis such that

$$\begin{cases} w_j \to w & \text{strongly in } L^p(\Omega), \\ \nabla w_j \rightharpoonup \nabla w & \text{weakly in } L^p(\Omega). \end{cases}$$

The continuity of the norm gives

$$\|w\|_{L^p(\Omega)} = \lim_{j\to\infty} \|w_j\|_{L^p(\Omega)} = 1.$$

On the other hand, the weak semicontinuity of the norm yields

$$\|\nabla w\|_{L^p(\Omega)} \le \liminf_{j\to\infty} \|\nabla w_j\|_{L^p(\Omega)} = 0$$

so that $\nabla w = 0$. Since Ω is connected, w is constant and since w satisfies one of the hypotheses (i), (ii), (iii), we infer $w = 0$, in contradiction to $\|w\|_{L^p(\Omega)} = 1$. $\square$

Corollary. *If $u \in W^{1,p}(\Omega)$, let*

$$\frac{1}{|\Omega|}\int_\Omega u\,dx = u_\Omega.$$

Then

$$\|u - u_\Omega\|_{L^p(\Omega)} \le C_P \|\nabla u\|_{L^p(\Omega)}.$$