

Hölder Spaces

Definition. (i) If $u : \Omega \rightarrow \mathbb{R}$ is bounded and continuous, we write

$$\|u\|_{C(U)} = \sup_{x \in U} |u(x)|.$$

(ii) The γ -th Hölder seminorm of $u : \Omega \rightarrow \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\overline{\Omega})} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\gamma},$$

and the γ -th Hölder norm is

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} := \|u\|_{C(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})}.$$

Definition. The Hölder space $C^{k,\gamma}(\overline{\Omega})$ consists of all functions $u \in C^k(\overline{\Omega})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\overline{\Omega})} + \sum_{|\alpha|=k} [u]_{C^{0,\gamma}(\overline{\Omega})}$$

is finite.

The space $C^{k,\gamma}(\overline{\Omega})$ consists of those functions u which are k -times continuously differentiable and whose k -th partial derivatives are bounded and Hölder continuous with exponent γ .

Theorem. The space of functions $C^{k,\gamma}(\overline{\Omega})$ is a Banach space.

Morrey's inequality

Suppose $n < p < \infty$. We shall show that if $u \in W^{1,p}(U)$, then u is in fact Hölder continuous after possibly redefined on a set of measure zero.

Lemma 1. There exists a constant C , depending only on n and r , such that

$$(1) \quad \frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y) - u(x)| dy \leq C \int_{B_r(x)} \frac{|Du(y)|}{|y - x|^{n-1}} dy, \quad \forall u \in C^1(\mathbb{R}^n).$$

Proof. Fix any point $w \in \partial B_1(0)$. Then if $0 < s < r$,

$$\begin{aligned} |u(x + sw) - u(x)| &= \left| \int_0^s \frac{d}{dt} u(x + tw) dt \right| \\ &= \left| \int_0^s Du(x + tw) \cdot w dt \right| \\ &\leq \int_0^s |Du(x + tw)| dt. \end{aligned}$$

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Hence

$$\begin{aligned} \int_{\partial B_1(0)} |u(x + s\omega) - u(x)| d\omega &\leq \int_0^s \int_{\partial B_1(0)} |Du(x + t\omega)| d\omega dt \\ &\leq \int_0^s \int_{\partial B_1(0)} |Du(x + t\omega)| \frac{t^{n-1}}{t^{n-1}} d\omega dt. \end{aligned}$$

Let $y = x + tw$, so that $t = |x - y|$. Then we have

$$\begin{aligned} \int_{\partial B_1(0)} |u(x + s\omega) - u(x)| d\omega &\leq \int_{B_s(x)} \frac{|Du(y)|}{|x - y|^{n-1}} dy \\ &\leq \int_{B_r(x)} \frac{|Du(y)|}{|x - y|^{n-1}} dy. \end{aligned}$$

Then

$$\begin{aligned} \int_{B_r(x)} |u(y) - u(x)| dy &= \int_0^r s^{n-1} \int_{\partial B_1(0)} |u(x + s\omega) - u(x)| d\omega ds \\ &\leq \int_0^r s^{n-1} \int_{B_r(x)} \frac{|Du(y)|}{|x - y|^{n-1}} dy ds \\ &= \frac{r^n}{n} \int_{B_r(x)} \frac{|Du(y)|}{|x - y|^{n-1}} dy. \quad \square \end{aligned}$$

Proposition 2. Assume $n < p < \infty$. Then there exists a constant C , depending only on p and n , such that

$$\sup_{\mathbb{R}^n} |u| \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall u \in C^1(\mathbb{R}^n).$$

Proof. We have

$$\begin{aligned} |u(x)| |B_1(x)| &= \int_{B_1(x)} |u(x)| dy \\ (2) \quad &\leq \int_{B_1(x)} |u(x) - u(y)| dy + \int_{B_1(x)} |u(y)| dy \\ &\leq \frac{C}{n} \int_{B_r(x)} \frac{|Du(y)|}{|x - y|^{n-1}} dy + |B_1(x)|^{\frac{p-1}{p}} \|u\|_{L^p(B_1(x))}, \end{aligned}$$

by (1) and Hölder's inequality. Further, Hölder's inequality yields

$$(3) \quad \int_{B_r(x)} \frac{|Du(y)|}{|x - y|^{n-1}} dy \leq \|Du\|_{L^p(\mathbb{R}^n)} \left(\int_{B_1(x)} \frac{1}{|x - y|^{(n-1)p/(p-1)}} dy \right)^{\frac{p-1}{p}} \leq C \|u\|_{W^{1,p}(\Omega)},$$

since $p > n$ implies $(n-1)\frac{p}{p-1} < n$, so that $\int_{B_1(x)} \frac{1}{|x - y|^{(n-1)p/(p-1)}} dy < \infty$. Substituting (3) into (2) completes the proof. \square

Theorem 3. Let Ω be a bounded, open and C^1 -domain in \mathbb{R}^n . Assume $n < p < \infty$ and $u \in W^{1,p}(\Omega)$. Then there exists a function $u^* \in C^0(\overline{\Omega})$, such that $u = u^*$ a.e. and there holds, for a constant C depending only on p , n and Ω ,

$$(4) \quad \sup_{\Omega} |u| \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

In other words, if $n < p < \infty$, then $W^{1,p}(\Omega)$ is continuously embedded in $C^0(\overline{\Omega})$.

Proposition 4. Assume $n < p < \infty$. Then there exists a constant C , depending only on p and n , such that

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall u \in C^1(\mathbb{R}^n).$$

Proof. Choose any two points $x, y \in \mathbb{R}^n$ and write $r = |x - y|$. Let $W = B_r(x) \cap B_r(y)$. Then

$$(5) \quad |u(x) - u(y)| |W| = \int_W |u(x) - u(y)| dz \leq \int_{B_r(x)} |u(x) - u(z)| dz + \int_{B_r(y)} |u(y) - u(z)| dz.$$

By (1) and Hölder's inequality,

$$(6) \quad \begin{aligned} \int_{B_r(x)} |u(x) - u(z)| dz &\leq \frac{1}{n} \int_{B_r(x)} \frac{|Du(y)|}{|x - y|^{n-1}} dy \\ &\leq \frac{1}{n} \|Du\|_{L^p(\mathbb{R}^n)} \left(\int_{B_r(x)} \frac{1}{|x - z|^{(n-1)p/(p-1)}} dz \right)^{\frac{p-1}{p}} \\ &\leq \frac{1}{n} \|Du\|_{L^p(\mathbb{R}^n)} \left(\omega_n \int_0^r r^{\frac{(n-1)p}{p-1} - (n-1)} dr \right)^{\frac{p-1}{p}} \\ &= C (r^{n - \frac{p(n-1)}{p-1}})^{\frac{p-1}{p}} \|Du\|_{L^p(\mathbb{R}^n)} = Cr^{1 - \frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Likewise,

$$(7) \quad \int_{B_r(y)} |u(y) - u(z)| dz \leq Cr^{1 - \frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}.$$

Substituting (4) and (5) into (3), we obtain

$$|u(x) - u(y)| \leq Cr^{1 - \frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)} = C|x - y|^{1 - \frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}.$$

Thus

$$[u]_{C^{0,1-(n/p)}(\mathbb{R}^n)} = \sup_{x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{1-(n/p)}} \right\} \leq C \|Du\|_{L^p(\mathbb{R}^n)}.$$

This and (3) completes the proof. \square

We finally arrive at the following.

Theorem 5 (Morrey's inequality). *Let Ω be a bounded, open, C^1 -domain in \mathbb{R}^n . Assume $n < p < \infty$ and $u \in W^{1,p}(\Omega)$. Then there exists a function $u^* \in C^{0,1-\frac{n}{p}}(\overline{\Omega})$, such that $u = u^*$ a.e. and there holds, for a constant C , depending only on p , n and Ω ,*

$$\|u\|_{C^{0,1-\frac{n}{p}}(\overline{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

Proof. Since $\partial\Omega \in C^1$, there exists an extension $Eu \in W^{1,p}(\mathbb{R}^n)$ such that

$$(8) \quad \begin{cases} Eu = u & \text{in } \Omega \\ Eu \text{ has compact support, and} \\ \|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}. \end{cases}$$

Since Eu has compact support, there exists a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that

$$(9) \quad u_m \rightarrow Eu \quad \text{in } W^{1,p}(\mathbb{R}^n),$$

and

$$(10) \quad \|u_m\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C\|u_m\|_{W^{1,p}(\mathbb{R}^n)}.$$

According to Theorem 4,

$$\|u_m - u_\ell\|_{C^{0,1-(n/p)}(\mathbb{R}^n)} \leq C\|u_m - u_\ell\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall \ell, m \geq 1;$$

hence, there exists a function $u^* \in C^{0,1-(n/p)}(\mathbb{R}^n)$ such that

$$(11) \quad u_m \rightarrow u^* \quad \text{in } C^{0,1-(n/p)}(\mathbb{R}^n).$$

In view of (9) and (11), we see that $u^* = u$ a.e. on Ω , i.e. u^* is a version of u . This and (9) yields

$$\|u^*\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C\|Eu\|_{W^{1,p}(\mathbb{R}^n)},$$

which and (8) completes the proof. \square

Sobolev Embedding Theorem.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^k -domain. If $k > \frac{n}{p}$, the Sobolev space $W^{k,p}(\Omega)$ is continuously embedded in $C^{k-[\frac{n}{p}]-1,\gamma}(\overline{\Omega})$, where*

$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer,} \\ \text{any positive number } < 1, & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

That is, there exists a number C depending only on k , p , n and Ω such that

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\overline{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)}, \quad \forall f \in W^{1,p}(\Omega).$$

Proof. (i) Assume that $\frac{n}{p}$ is not an integer. Set

$$\ell = \left[\frac{n}{p} \right], \quad \text{i.e. } \ell < \frac{n}{p} < \ell + 1.$$

Then, since $\ell p < n$, by Sobolev embedding theorem*,

$$(11) \quad u \in W^{k-\ell, r}(\Omega), \quad \text{where } \frac{1}{r} = \frac{1}{p} - \frac{\ell}{n}.$$

Observe that $r = \frac{pn}{n-p\ell} > n$. Hence (11) and Morrey's inequality imply that

$$D^\alpha u \in C^{0, 1-\frac{n}{\gamma}}(\overline{\Omega}), \quad \forall |\alpha| \leq k - \ell - 1.$$

Observe also that $1 - \frac{n}{r} = 1 - \frac{n}{p} + \ell = \left[\frac{n}{p}\right] + 1 - \frac{n}{p}$. Thus

$$u \in C^{k - \left[\frac{n}{p}\right] - 1, \left[\frac{n}{p}\right] + 1 - \frac{n}{p}}(\overline{\Omega}).$$

(ii) Suppose $\frac{n}{p}$ is an integer. Set $\ell = \left[\frac{n}{p}\right] - 1 = \frac{n}{p} - 1$.

- We have as above $u \in W^{k-\ell, r}(\Omega)$, for $r = \frac{pn}{n-p\ell} = n$.
- Hence, the Sobolev embedding theorem* shows that

$$D^\alpha u \in L^q(\Omega), \quad \forall n \leq q < \infty \quad \text{and} \quad \forall |\alpha| \leq k - \ell - 1 = k - \left[\frac{n}{p}\right].$$

Therefore Morrey's inequality further implies

$$D^\alpha u \in C^{0, 1-\frac{n}{q}}(\overline{\Omega}), \quad \forall n < q < \infty \quad \text{and} \quad \forall |\alpha| \leq k - \left[\frac{n}{p}\right] - 1.$$

Consequently, $u \in C^{k - \left[\frac{n}{p}\right] - 1, \gamma}(\overline{\Omega})$ for each $0 < \gamma < 1$. \square