Stellar Physics
lecture 1: basic concepts and equations

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Hydrostatic equilibrium

a spherically symmetric star

$$\rho\ddot{r} = -\frac{dp}{dr} - \rho g, \quad g = \frac{Gm}{r^2} = \frac{G\left(\int_0^r 4\pi r^2 \rho dr\right)}{r^2}$$

hydrostatic equilibrium: $\rho g = -\dfrac{dp}{dr}$

recall: if $\rho g > -dp/dr \Rightarrow \left(\dfrac{R^3}{GM}\right)^{1/2} \approx \dfrac{1}{\sqrt{G\rho}} (= t_{ff}) < \dfrac{R}{c_s} (= t_{sound\ crossing}),$

then Jeans unstable.

In hydrostatic equilibrium: $t_{ff} \approx t_{sound\ crossing}$

Commonly, they are called the dynamical timescale, describing how long it takes for an entire star to restore/adjust to the hydrostatic equilibrium in response to a perturbation. For the sun, t_{dyn} ~ 30 mins.

N.B.: energy equation, which is related the “thermal timescale”, is not involved

Virial theorem

The force equation in the last slide describes the local balance of momentum at all points in the star. The virial theorem, by contrast, representing a global, or integrated, expression of momentum balance.

fluid: Eulerian description (r,t) & Lagrangian description (m,t)
where m is the mass coordinate

$$\rho\ddot{r} = -\frac{\partial p}{\partial r} - \rho\frac{Gm}{r^2} \Rightarrow \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} = -\frac{\partial p}{\partial m} - \frac{Gm}{4\pi r^4}$$

$$\Rightarrow \frac{1}{2}\frac{\partial^2 I}{\partial t^2}(\text{virial}) \equiv \frac{1}{2}\frac{\partial}{\partial t^2}\int_0^M r^2 dm = \int_0^M \dot{r}^2 dm + 3\int_0^M \frac{p}{\rho}dm - \int_0^M \frac{Gm}{r}dm,$$

where $\partial m = \rho 4\pi r^2 \partial r$.

Assume an ideal gas $p = nkT = (R/\mu)\rho T$ with $R = k/m_u = 8.31\times10^7\,\text{erg K}^{-1}\,\text{g}^{-1}$

and $\rho = nm_{partile} = n\mu m_u$ and $m_u = 1\,\text{amu} = 1.66\times10^{-24}\,\text{g}$,

$$\frac{p}{\rho} = \frac{R}{\mu}T = (c_p - c_v)T = (\gamma-1)c_v T = (\gamma-1)u, \quad \gamma \equiv c_p/c_v$$

$$\Rightarrow \frac{1}{2}\frac{d^2 I}{dt^2} = 2E_{kinetic} + 3(\gamma-1)E_i + E_g = 2E_{kinetic} + 2E_i + E_g \quad \text{for a monatomic gas } \gamma = 5/3$$

In hydrostatic equilibrium, $\dot{r} = \ddot{r} = 0 \Rightarrow 2E_i + E_g = 0$

Virial theorem

For a general equation of state, define a quantity ζ by $3\dfrac{p}{\rho} \equiv \zeta u$

an ideal gas: $\zeta = 3(\gamma - 1)$, $\zeta = 2$ for the monatomic case.

a photon gas: $p = aT^4/3$ and $\rho u = aT^4 \Rightarrow \zeta = 1$

If ζ is constant throughout the star, the general virial theorem in hydrostatic equilibrium is given by $\zeta E_i + E_g = 0$, i.e.

E_i and E_g are coupled in a gravitational bound system.

Full version of the virial theorem includes more terms:
kinetic energy, magnetic energy, surface pressure & magnetic terms, and even in a tensor form.

Recall: virial theorem applies to any gravitational bound systems, such as planetary system, stellar cluster, cluster of galaxy. One can obtain the so-called "virial mass" of a system by temperature (M_{virial}~$3kTR/Gm_u$) or by velocity dispersion (M_{virial}~$\sigma^2 R/2G$), which gives an upper limit of the true mass of the system.

negative specific heat

Define the total energy

$$W = E_i + E_g = (1-\zeta)E_i = \frac{\zeta - 1}{\zeta} E_g,$$

$W < 0$ for a gravitationally bound system.

$W = 0$ in the case of $\zeta = 1$ $(\gamma = 4/3)$.

$-W$ is the binding energy of the system.

Let L be the luminosity of the star. Conservation of energy demands

$$L = -\frac{dW}{dt} = (\zeta - 1)\frac{dE_i}{dt} = -\frac{\zeta - 1}{\zeta}\frac{dE_g}{dt}$$

For an ideal gas undergoing gravitational contraction $(\dot{E}_g < 0)$:

$$L = -\dot{E}_g / 2 = \dot{E}_i$$

i.e. half of the energy liberated by the contraction is radiated away and the other half is used to heat the star → behave like a body having a negative specific heat.

Recall: gravothermal catastrophe of stellar clusters

Can gravitational energy power a star?

The Kelvin-Helmholtz time-scale for a cooling and hence contracting star:

$$t_{KH} = \frac{|E_g|}{L} \approx \frac{E_i}{L} \approx \frac{GM^2}{RL}$$

$$\text{For the Sun, } L = 3.8\times10^{33} \ \text{erg/s}$$

$$\Rightarrow t_{KH} \approx 1.6\times10^{7} \ \text{years} << \ t_{Sun} \approx 4.6\times10^{9} \ \text{years!}$$

call for another energy source: nuclear energy,
binding energy per nucleon rises to a maximum near A=56

But, we shall see that gravitational contraction is the main stellar energy source during the pre-main sequence phase. Then a proto-star evolves on the time-scale t_{KH}.

Nuclear energy timescale

4 Hydrogen nuclei: 6.693×10^{-27} kg
1 Helium nucleus: 6.645×10^{-27} kg
mass difference = 0.7 %
Einstein energy-mass equivalent relation $E=\Delta mc^2$

$$4\mathrm{H} \rightarrow \mathrm{He} + Q, \quad Q = 6.3\times10^{18}\,\mathrm{erg/g}$$

$$t_{nuc} \equiv \frac{E_{nuc}}{L} \approx \frac{QM_{sun}}{L_{sun}} \approx 10^{11} \ \ \mathrm{years}$$

Energy equation for stars

The first law of thermodynamics:

$$dq = du + pdv = du + pd(1/\rho) \Rightarrow \varepsilon - \frac{dl}{dm} = \frac{du}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt}$$

where ε is the heating rate per unit mass (nuclear burning, tidal heating, viscous heating, etc. depending on the problem) and $l = 4\pi r^2 F$ is the luminosity rate passing outward through a sphere of mass coordinate m. $l = 0$ at $m = 0$ due to symmetry and $= L$ at $m = M$.

Energy equations for stars

In hydrostatic equilibrium, the pdv work is related to the global gravitational energy :

$$E_g = -3\int_0^M \frac{p}{\rho} dm \Rightarrow \dot{E}_g = -3\int_0^M \frac{\dot{p}}{\rho} dm + 3\int_0^M \frac{p}{\rho^2} \dot{\rho} dm$$

$$\dot{E}_g = \int_0^M \frac{Gm}{r} \frac{\dot{r}}{r} dm \text{ together with } \frac{\partial \dot{p}}{\partial m} = 4\frac{Gm}{4\pi r^4} \frac{\dot{r}}{r} \Rightarrow \dot{E}_g = -\frac{3}{4}\int_0^M \frac{\dot{p}}{\rho} dm$$

The above two give $\dot{E}_g = -\int_0^M \frac{p}{\rho^2} \dot{\rho} dm$

So we have the global energy conservation in hydrostatic equilibrium by integrating the local energy conservation over m :

$$\dot{E}_{nuc} - L - L_\nu = \dot{E}_i + \dot{E}_g \Rightarrow \dot{E}_{nuc} - L - L_\nu = \frac{\zeta - 1}{\zeta} \dot{E}_g. \text{ thermal expansion/contraction}$$

If hydrostatic condition does not apply, we would have

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(E_{nuc} - E_i - E_g - E_{kinetic}\right) - L - L_\nu = 0$$

Time scales

$$t_{nuc} >> t_{KH} >> t_{dyn}$$

In most cases, timescales under consideration >> t_{dyn}, hydrostatic is a good approximation.

If we consider a process with timescale >> t_{KH}, such as the stellar evolution governed by nuclear fusion, the adjustment to new hydrostatic and thermal equilibrium ($\varepsilon = dl/dm$) should be quickly reached and therefore the hydrostatic and thermal equilibrium (i.e. time derivative terms in the momentum and energy equations can be set to zero) can be used for model calculation.

If we consider a process with timescale << t_{KH}, such as some pulsating stars, the change of stellar properties is nearly adiabatic, i.e. heating and cooling processes can be ignored.

summary

$$\text{hydrostatic equilibrium}: \; -\frac{\partial p}{\partial m} = \frac{Gm}{4\pi r^4}$$

$$\text{mass conservation}: \; \frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$$

$$\text{energy conservatoin}: \; \varepsilon - \frac{dl}{dm} = \frac{du}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt}$$

Global descriptions of force and energy balance:

virial theorem: in hydrostatic equilibrium, E_i and E_g are coupled

global energy: describe thermal evolution, gives a more intuitive view for energy transfer between different forms