Stellar Physics
lecture 2: energy transport

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Radiation intensity and flux

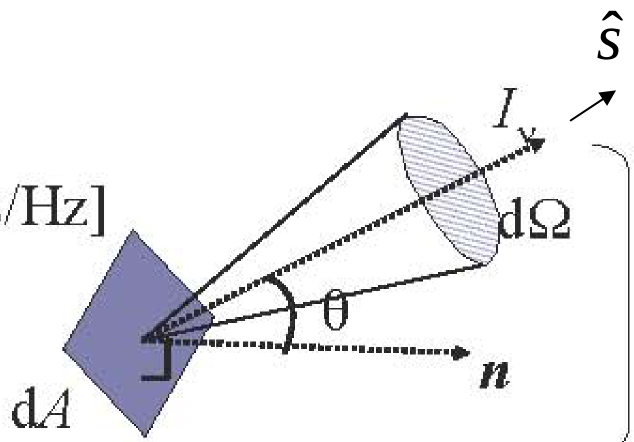
Definition of Intensity I_ν :

$$dE_\nu(t) = I_\nu(\vec{r}, \hat{s}, t)\, d\Omega\, dA\, d\nu\, dt$$

I_ν [erg/cm^2/s/sr/Hz]

Definition of **Flux** F_ν [erg/cm^2/s/Hz]

$$F_\nu = \oint I_\nu \cos\theta \, \mathrm{d}\Omega$$



Radiative transfer equation

e.g. Rybicki & Lightman:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \Rightarrow \frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

$$d\tau_\nu = \rho\kappa_\nu ds, \quad l_{ph} = \frac{1}{\rho\kappa_\nu} = \frac{1}{n\sigma_\nu}$$

For the sun, $\bar{\rho} \approx 1.4\ \mathrm{g\ cm^{-3}}$, $\kappa \approx 0.4\text{-}1\ \mathrm{cm^2\ g^{-1}}$

$\Rightarrow l_{ph} \approx 2\ \mathrm{cm} \Rightarrow$ stellar matter is very opaque.

Blackbody radiation

■ a black body is a perfect radiator that absorbs all radiation incident on it (reflects no light, and so-named as blackbody) and reemits radiation in a frequency spectrum depending only on its temperature T. Blackbody radiation is isotropic, and its power spectrum is described by the Planck function $B_\nu(T)$

■ thermal radiation is radiation emitted by matter in thermal equilibrium. Thermal radiation becomes blackbody radiation only for optically thick media ($\tau = \int \alpha \, dr \gg 1$) and in this case $I_\nu = B_\nu$.

■ In a fluid with a temperature gradient, we may apply the concept of "local thermodynamic equilibrium (LTE) as a local version of complete thermodynamic equilibrium. The LTE is valid when collisional processes dominate over radative processes, hence enabling matter and radiation to share the same temperature.

■ Kirchhoff's Law: under LTE, $S_\nu = B_\nu$

$$B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{\exp(h\nu / kT) - 1}$$

Radiative diffusion

radiative tranfer equation (e.g. Rybicki & Lightman):

$l_{ph} << R_* \Rightarrow$ adopt the plane-parallel assumption

$$\cos\theta \frac{dI_\nu}{dr} = \frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

$$\Rightarrow \cos\theta \frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

In the optically thick regim, LTE $\Rightarrow S_\nu = B_\nu$ and I_ν changes slowly over $\frac{1}{\rho\kappa}$.

Hence the derivative term is small and we can solve for I_ν using iteration.

$$I_\nu^{(0)} \approx S_\nu^{(0)} = B_\nu$$

$$\Rightarrow I_\nu^{(1)} \approx B_\nu - \cos\theta \frac{dB_\nu}{dr}, \quad I_\nu^{(2)} = B_\nu - \cos\theta \frac{dI_\nu^{(1)}}{dr}, \ldots\ldots\ldots\ldots$$

Consider 1st order, $F_\nu = \int \cos\theta \, I_\nu^{(1)} d\Omega = -\frac{4\pi}{3\kappa_\nu \rho} \frac{dB_\nu}{dT} \frac{dT}{dr}$

Then, $F_{rad} = \int_0^\infty F_\nu d\nu = -\frac{16\sigma T^3}{3\kappa_R \rho} \frac{dT}{dr} = \frac{16\sigma T^4}{3\kappa_R \rho l_p} \nabla,$

where $\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} d\nu}{\int_0^\infty \frac{dB_\nu}{dT} d\nu}$ and $\nabla \equiv \frac{d \ln T}{d \ln p}$.

Radiative diffusion

The Rosseland mean opacity is the average of $1/\kappa_\nu$. That is, more Energy is transported at frequencies where the matter is more transparent. It is also a mean weighted with *dB/dT*; this means that more energy is transported at frequencies where the radiation field is more temperature dependent.

Convective instability

e.g. Frank Shu: gas dynamics

$$d\rho_{blob} = \left(\frac{\partial \rho}{\partial p}\right)_s dp, \quad d\rho_{ambient} = \left(\frac{\partial \rho}{\partial p}\right)_s dp + \left(\frac{\partial \rho}{\partial s}\right)_p ds$$

Convective instability will arise if

$d\rho_{blob} < (>) d\rho_{ambient}$ for a upward (downward) displacement

or

$\left(\frac{\partial \rho}{\partial s}\right)_p ds > (<) 0$ for a upward (downward) displacement

Now, Maxwell's relations give

$$\left(\frac{\partial \rho^{-1}}{\partial s}\right)_p = \left(\frac{\partial T}{\partial p}\right)_s > 0$$

$\Rightarrow \frac{ds}{dr} < 0$ for convective linear instability (Schwarzchild's criterion)

Nonlinear outcome is that strong convective mixing would smear out the entropy gradient.

Convective instability

the criterion can be expressed by temperature gradient :

1st law of thermal dynamics applies to an ideal gas

$$ds = \frac{du}{T} + \frac{p}{T} d\rho^{-1} = \frac{du}{dT} d\ln T - \frac{R}{\mu} d\ln\rho = \frac{du}{dT}\left(d\ln p - d\ln\rho\right) - \frac{R}{\mu} d\ln\rho$$

$$= c_v(d\ln p - d\ln\rho) - (c_p - c_v) d\ln\rho$$

$$\Rightarrow ds = c_v d\ln(p/\rho^{\gamma}) = c_v d\ln(T^{\gamma}/p^{\gamma-1}) = c_v \gamma d\ln p \left(\frac{d\ln T}{d\ln p} - \frac{\gamma-1}{\gamma}\right)$$

$$\Rightarrow \frac{ds}{dr} = -c_p \frac{1}{l_p}\left[\frac{d\ln T}{d\ln p} - \left(\frac{d\ln T}{d\ln p}\right)_{ad}\right] \equiv -\frac{c_p}{l_p}\left(\nabla - \nabla_{ad}\right)$$

If $\nabla > \nabla_{ad}$, convective instability occurs. Efficient convection (e.g. fully convective stars or brown dwarfs) tends to give rise to the adiabatic temperature gradient.

Brunt-Väisälä (buoyancy) frequency

$\Delta r \propto \exp(iNt)$: small displacement of the blob with a timescale described by $1/N$.

$$\frac{d^2\Delta r}{dt^2} = \frac{\delta\rho}{\rho_{ambient}} g = \frac{g}{\rho_{ambient}} \left(\frac{d\rho_{blob}}{dr} - \frac{d\rho_{ambient}}{dr} \right) \Delta r = -\frac{\mathrm{g}}{\rho} \left(\frac{\partial\rho}{\partial s} \right)_p \frac{ds}{dr} \Delta r$$

$$N^2 = g\rho \left(\frac{\partial T}{\partial p} \right)_s \frac{ds}{dr} = g \frac{\rho T}{p} \frac{\gamma - 1}{\gamma} c_v \frac{d(\ln p/\rho^\gamma)}{dr} = g \frac{d \ln\left(\frac{p^{1/\gamma}}{\rho} \right)}{dr} = \frac{g}{l_s},$$

where $l_s = (d \ln s / dr)^{-1}$ is the entropy scale height.

When $l_s < 0$, $N^2 < 0$ and therefore the convective instability grows at a rate of $|N|$. The fluid is convectively unstable.

When $l_s > 0$, $N^2 > 0$ and the blob oscillates at the Brunt - Vaisala (buoyancy) freq N (i.e. eigenfrequency of a stably stratified fluid). No convection occurs.

Mixing-length theory for convection

Convection is one type of turbulence driven by an entropy gradient against gravity. How to model the energy transport by turbulent mixing? The turbulence cascade theory describes how the eddy kinetic energy is passed from the large-scale eddies to the microscopic scale where the energy is dissipated. Thermal convection transports heat from high-temperature inner region to the outer part of a star, which is a non-linear (i.e. *non-local*) process.

The mixing-length theory is a phenomenological model describing a simple picture of convection in analogy to molecular heat transfer: the transporting “particles” are macroscopic mass element (“blobs”) instead of molecules. The “blob” travels over a distance called the mixing length, Λ, before mixing with the surroundings and hence transporting energy. Based on the theory, the convective flux may be written in terms of *local* fluid quantities as follows

$$F_{conv,MLT} = \rho \mathrm{v}_{conv} \left| \frac{Tds}{dr} \right| \Lambda \quad \text{with} \quad \Lambda = \alpha l_p$$

parametrized in terms of pressure scale height & one single isotropic eddy

Mixing-length theory for convection

$$\mathrm{v}_{\mathrm{conv}} = \frac{1}{2\sqrt{2}}|N|\Lambda = \Lambda\sqrt{\frac{g}{l_p}(\nabla - \nabla_{ad})}, \quad \frac{1}{2\sqrt{2}} \text{ is from numerical simulation}$$

$$F_{conv} = \rho \mathrm{v}_{conv} T \left|\frac{ds}{dr}\right| \Lambda = \rho \mathrm{v}_{conv} T c_p \frac{\Lambda}{l_p}(\nabla - \nabla_{ad})$$

$$= \frac{1}{2\sqrt{2}} \rho c_p T \sqrt{g} \Lambda^2 \left(\frac{\nabla - \nabla_{ad}}{l_p}\right)^{3/2}$$

Total energy flux

$$F = F_{rad} + F_{conv}$$

$$F_{rad} = \frac{4acT^4}{3\rho\kappa_R l_p}\nabla$$

$$F \equiv \frac{4acT^4}{3\rho\kappa_R l_p}\nabla_{rad} \quad (\nabla_{rad} \text{ is related to the total flux})$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho}{4acT^3}F_{rad} = -\frac{3\kappa\rho}{4acT^3}\left(\frac{\nabla}{\nabla_{rad}}\right)F$$

Efficiency of convection

reference: Cox & Giuli 1968, Principles of stellar structure

Convective blobs can exchange heat with ambient gas via radiation, which reduces convective flux and damps convective motion. The thermal properties of the blobs are not adiabatic. In other words,

$$\nabla_{\mathrm{ad}} \ \text{should be replaced by} \ \nabla' \equiv \left.\frac{d\ln T}{d\ln p}\right|_{\text{convective blob}}$$

$$\text{Note that in the convection zone}: \nabla_{rad} > \nabla > \nabla' > \nabla_{ad}$$

recall:

$$\nabla \equiv \frac{d\ln T}{d\ln P}$$

$$\nabla_{ad} \equiv \left.\frac{d\ln T}{d\ln P}\right|_{ad}$$

$$\nabla' \equiv \left.\frac{d\ln T}{d\ln P}\right|_{convective\ blob}$$

Therefore,

$$\mathrm{v}_{\mathrm{conv}} = \frac{1}{2\sqrt{2}}|N|\Lambda = \Lambda\sqrt{\frac{g}{l_p}(\nabla - \nabla')}$$

$$F_{conv} = \rho \mathrm{v}_{conv} T \left|\frac{ds}{dr}\right| \Lambda = \rho \mathrm{v}_{conv} T c_p \frac{\Lambda}{l_p}(\nabla - \nabla') = \frac{1}{2\sqrt{2}} \rho c_p T \sqrt{g} \Lambda^2 \left(\frac{\nabla - \nabla'}{l_p}\right)^{3/2}$$

Efficiency of convection

"Stellar structure and evolution" by Kippenhahn & Weigert

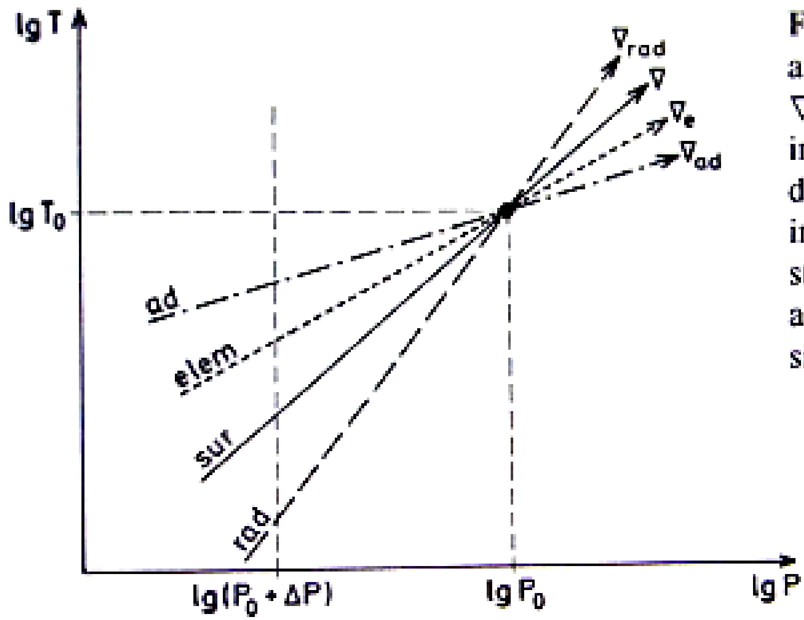


Fig. 6.2. Temperature–pressure diagram with a schematic sketch of the different gradients $\nabla(\equiv \partial \ln T / \partial \ln P)$ in a convective layer. Starting at a common point with P_0 and T_0, the different types of changes (adiabatic, in a rising element, in the surroundings, for radiative stratification) lead to different temperatures at a slightly higher point with $P_0 + \Delta P$ $(< P_0$, since P decreases outwards)

Efficiency of convection

How to calculate ∇' and hence F_{conv}?

Define

$$\Gamma \equiv \frac{\text{energy carried by convection just before dissolving}}{\text{energy loss via radiation during lifetime}}$$

$$= \frac{c_p \rho T_{\max} V_{eddy}}{\mathrm{A}(4ac/3)T^3 \Delta T(\Lambda / v_{conv})/(\kappa\rho)/(\Lambda/2)}$$

$$= \frac{c_p}{6ac}\frac{\kappa\rho^2 v_{conv}\Lambda}{T^3},$$

where numerical values $T_{\max} = 2\Delta T$, $V_{eddy}/A = (2/9)\Lambda$ have been used.

In addition, by definition

$$\Gamma = \frac{F_{conv,non-ad}}{F_{conv,ad} - F_{conv,non}} = \frac{\nabla - \nabla'}{(\nabla - \nabla_{ad}) - (\nabla - \nabla')} = \frac{\nabla - \nabla'}{\nabla' - \nabla_{ad}}$$

So, ∇' is solved from the above two equations for Γ in terms of ∇ and ∇_{ad}.

now, $F = F_{rad} + F_{conv} \Rightarrow \nabla_{rad} = \nabla + (9/2)\Gamma(\nabla - \nabla')$

i.e. ∇_{rad} can be expressed in terms of ∇ and ∇_{ad}

Equations for stellar structure

$$\text{hydrostatic equilibrium}: \ -\frac{\partial p}{\partial m} = \frac{Gm}{4\pi r^4}$$

$$\text{mass conservation}: \ \frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$$

$$\text{energy conservatoin}: \ \varepsilon - \frac{dl}{dm} = \frac{du}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt}$$

$$\text{radiatve diffusion}: \frac{dT}{dm} = -\frac{3\kappa_R}{16ac\pi r^2 T^3} F_{rad} = -\frac{3\kappa_R}{16ac\pi r^2 T^3}\left(\frac{\nabla}{\nabla_{rad}}\right)F$$

$$\text{equation of state}: \ \rho(p,T)$$

5 equations for 5 unknowns: p, ρ, T, F, r as functions of (m,t).

need initial and boundary conditions

Boundary conditions

$$F_\nu = -\frac{4\pi}{3}\frac{1}{\kappa_\nu \rho}\frac{dB_\nu}{dr}, \; p_{rad,\nu} = \frac{u_\nu}{3} = \frac{4\pi}{3c}B_\nu$$

$$\Rightarrow \frac{c}{\rho}\frac{dp_{rad,\nu}}{dr} = -\frac{\kappa_\nu \rho F_\nu}{c}. \text{ Integrate it over frequency}$$

$$\Rightarrow \frac{dp_{rad}}{dr} = -\frac{\kappa\rho}{c}\frac{L}{4\pi r^2} = -\frac{\kappa\rho F}{c} \quad \text{with} \quad \kappa = \frac{1}{F}\int_0^\infty \kappa_\nu L_\nu d\nu$$

$$\Rightarrow p_{rad} = \frac{F}{c}\tau + p_{rad}(0)$$

What is $p_{rad}(0)$? Consider only outgoing radiation at $\tau = 0$

$$p_{rad} = \frac{1}{c}\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi/2} I(0)\cos^2\theta d\Omega = \frac{2\pi}{3c}I(0)$$

$$F = \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi/2} I(0)\cos\theta d\Omega = \pi I(0)$$

The above two give $p_{rad}(0) = \frac{2}{3}\frac{F}{c}$.

Finally, using $p_r = \frac{1}{3}aT^4 = \frac{4\sigma}{3c}T^4$ in the optically thick limit, we have

$$\frac{4}{3}\frac{\sigma}{c}T^4 = \frac{F}{c}\tau + \frac{2}{3}\frac{F}{c} \Rightarrow T^4 = \frac{T_{eff}^4}{2}\left(1 + \frac{3}{2}\tau\right),$$

The photosphere is defined by $T = T_{eff} \Rightarrow \tau_p = \frac{2}{3}$

$$\frac{dp}{d\tau} = \frac{g}{\kappa} \Rightarrow p_p = g_p\int_0^{\tau_p}\frac{1}{\kappa}d\tau \Rightarrow p_p \approx \frac{2}{3}\frac{g_p}{\kappa_p}$$

The above boundary condition can be understaood by

$\kappa_p \rho_p l_p = 2/3.$

references:
Rybicki & Lightman
Hansen & Kawaler
Cox & Giuli

Boundary conditions

4 ODEs need 4 boundary conditions to solve:

$$m = 0: \quad r = 0, \quad F = 0 \ \text{(by spherical symmetry)}$$

$$m = M: \quad p_p = \frac{2}{3}\frac{g}{\kappa} \ (l_{mfp} \approx l_p), \quad F = \sigma T^4 \ \text{(blackbody)}$$

summary

- Equations for stellar structure and evolution
- Energy transport inside a star: radiative diffusion (Rosseland mean) & thermal convection (mixing length theory)
- Convection: Schwarzchild criterion, Brunt-Väisälä freq, efficiency
- Photospheric conditions