Stellar Physics
Lecture 4: composition, ionization, and opacity

Reference: Rybicki & Lightman

辜品高

gu@asiaa.sinica.edu.tw

Mean molecular weight

composition by mass fraction : $X_i, \ \sum_i X_i = 1$

$X \equiv X_H, Y \equiv X_{He}, Z = 1 - X - Y \equiv X_{\text{heavier than He}}$

$$n_i = \frac{\rho_i}{\mu_i m_u} = \frac{\rho \mathrm{X_i}}{m_u \mu_i}$$

If the gases are ideal and fully ionized,

$$p_{gas} = p_e + \sum_i p_i = \left(n_e + \sum_i n_i\right) kT = \sum_i (Z_i + 1) n_i kT = R \sum_i \frac{X_i (1 + Z_i)}{\mu_i} \rho T, \ \text{where} \ R = \frac{k}{m_u}.$$

mean molecular weight : $\mu \equiv \left(\sum_i \frac{X_i (Z_i + 1)}{\mu_i} \right)^{-1}.$

pure fully ionized hydrogen : $X_H = 1, \mu_H = 1, Z_H = 1 \Rightarrow \mu = 0.5$

neutral gas : $Z_i = 0$

If temperature is high, radiation pressure can't be ignored :

$$p = p_{gas} + p_{rad} = \frac{R}{\mu} \rho T + \frac{aT^4}{3}$$

define mean molecular weigth per free electron

$$\mu_e = \left(\sum_i X_i Z_i / \mu_i \right)^{-1}$$

$\mu_i / Z_i \approx 2$ for elements heavier than He

$$\Rightarrow \mu_e \approx \left(X + \frac{1}{2} Y + \frac{1}{2} (1 - X - Y) \right)^{-1} = \frac{2}{1 + X}$$

Thermal Ionization fraction: Saha eqn.

Mean molecular weight and hence thermodynamic properties depend on the composition and the degree of ionization.

The Boltzmann formula gives the ratio between the numbers in the two states of atoms in thermodynamical equilibrium:

$\frac{n_b}{n_a} = \frac{g_b}{g_a} \exp\left(-(E_b - E_a)/kT\right)$, where g_s is the statistical weight, namely, the degree of degeneracy of the level, or the number of quantum states all of which correspond to the s^{th} energy level. (N.B. the relative population of each level depends in a detailed way upon the mechanisms for populating and de-populating them: radiative, collision, & spontaneous. Recall that LTE happens when collisions dominate.)

Thermal Ionization fraction: Saha eqn.

The Boltzmann equation can be adapted to include states above the ionization potential of the atom. This gives the ratio of atoms in two different ionization states (e.g. ratio of HI to HII).

Let's think of a reaction of thermal ionizatoin and recombination $A_r \leftrightarrow A_{r+1} + e^-$

n_r : number density of an atom losing r electrons from the ground state.

dn_{r+1} : number density of an atom losing $r+1$ electrons from the ground state with the free electron in the momentum interval $[p_e, p_e + dp_e]$

Assume thermodynamic equilibrium and so the Boltzmann formula applies (i.e. ionization is balanced by recombination). Hence,

$\frac{dn_{r+1}}{n_r} = \frac{g_{r+1} dg(p_e)}{g_r} \exp\left(-\frac{\chi_r + p_e^2/2m_e}{kT}\right)$, where $dg(p_e) = \frac{2dVd^3 p_e}{h^3} = \frac{8\pi p_e^2 dp_e}{n_e h^3}$ is the statistical weight of the free electron in the momentum interval $[p_e, p_e + dp_e]$ (i.e. the number of phase space states available in volume V and momentum range p_e to $p_e + dp_e$ to a single electron when the electron density is n_e.)

$$\Rightarrow \frac{n_{r+1}}{n_r} = \frac{g_{r+1}}{g_r} \frac{8\pi}{n_e h^3} \exp(-\chi_r / kT) \int_0^\infty p_e^2 \exp\left(-\frac{p_e^2}{2m_e kT}\right) dp_e$$

using $\int_0^\infty x^2 \exp(-a^2 x^2) dx = \frac{\sqrt{\pi}}{4a^3}$, so we have the Saha equation

$$\frac{n_{r+1}}{n_r} n_e = 2 \frac{g_{r+1}}{g_r} \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2} \exp(-\chi_r / kT)$$

the Saha equation can be generalized to include ionization from all excited states by introducing the partition function.

In the literature, the Saha equation is normally written as

$$\frac{n_j^+ n_e}{n_j - n_j^+} = \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2} \exp(-\chi_j / kT),$$

where n_j and n_j^+ are the total and singly ionized number densities of constituent j respectively,

$n_e = \sum n_j^+$ is the total electron number density.

Thermal Ionization fraction: Saha eqn.

For example, for a gas of pure hydrogen:

$$n_e = n^+,\ n^+ + n_{neutral} = n$$

define the degree of ionization

$$x = \frac{n^+}{n} = \frac{n_e}{n}$$

$$\text{statistic weights}: \ g_{HII} = 1, \quad g_{HI} = 2n^2 \approx 2 \quad \left(\begin{array}{l} \because \ \text{most HI in the groud state } n = 1 \text{ if} \\ \text{the energy of 1st excitation state } 10.2\text{eV} >> kT \end{array} \right)$$

The Saha equation becomes (i.e. take $n_r = n_{neutral}$ and $n_{r+1} = n^+$)

$$\frac{x^2}{1-x} = \frac{1}{n}\left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{\chi_H}{kT}\right) = \frac{4.01\times10^{-6}}{\rho} T^{3/2} \exp\left(-\frac{1.578\times10^5}{T}\right)$$

We see that as $T \to \infty$ (0), $x \to 1$ (0)

It can be shown that in general $x \approx 0.5$, $\dfrac{\chi_\mathrm{H}}{kT} \sim 10$

i.e. the gas of hydrogen is partially ionzied when $T \approx 10^4$ K.

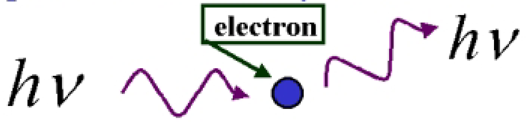
opacity

(1)Thomson (electron) scattering : when $h\nu < m_e c^2$, oscillating electron acts as a classic dipole

$$\kappa_{es} = \frac{8\pi}{3}\frac{r_e^2}{\mu_e m_u} = 0.2(1+X)\ \mathrm{cm^2 g^{-1}},\ \text{where } r_e = \frac{e^2}{m_e c^2} \text{ is the classical electron radius.}$$

independent of the frequency (but valid when $h\nu < m_e c^2$), ρ, and T.

low-E photon scattered by electron -



electron recoil can be ignored → elastic (coherent) scattering, cf. compton scattering when h ν > $m_e c^2$

(2) free - free (Bremsstrahlung) absorption

$$\kappa_\nu \propto (\text{probability} \propto \rho)(1/\text{thermal velocity} \propto T^{-1/2})\nu^{-3}$$

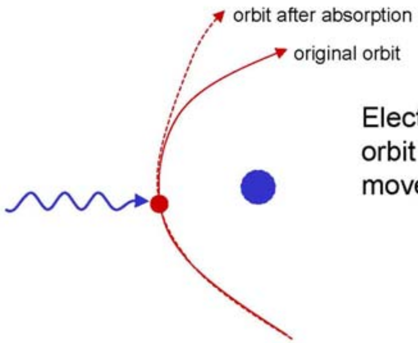
LTE : Kirchoff's law

$$S_\nu = \frac{j_\nu}{\rho\kappa_\nu} = B_\nu(T) \propto \nu^3$$

$$\Rightarrow \kappa_{R,ff} \approx 3.8\times 10^{22}\rho T^{-7/2}\quad \text{(Kramers opacity)}$$

Note that $\kappa_\nu \propto \nu^{-n} \Rightarrow \kappa_R \propto T^{-n}$.

continuum opacity



Electron moving in an unbound (hyperbolic) orbit about an ion absorbs a photon and moves into a higher energy orbit

opacity

(3) bound-free absorption (or photo-ionization): photon is absorbed by a bound electron, giving its energy above the ionization potential → continuum opacity

(4) bound-bound absorption: photon is absorbed by a bound system, exciting it to a higher energy state. Since the transitions are discrete, one would expect that absorption in a few lines gives only a small contribution. However, the absorption lines in stars are strongly broadened by collisions (Doppler broadening), which enhances its contribution to opacity.

(5) H^- opacity: there exists a bound state for a second electron in the field of a proton. This second electron is loosely bound – absorption of photons with h ν >0.75 eV (λ < 1655 nm, infrared), giving rise to a bound-free and free-free transitions.

$$H^- + \gamma \rightarrow H + e^-$$

$$H^- + e^- + \gamma \rightarrow H^- + e^-$$

Forming H^- requires Neutral H and free electrons. Free electrons can be from existing ionized H or from alkali metals (Na, K….) which have low ionization potentials.

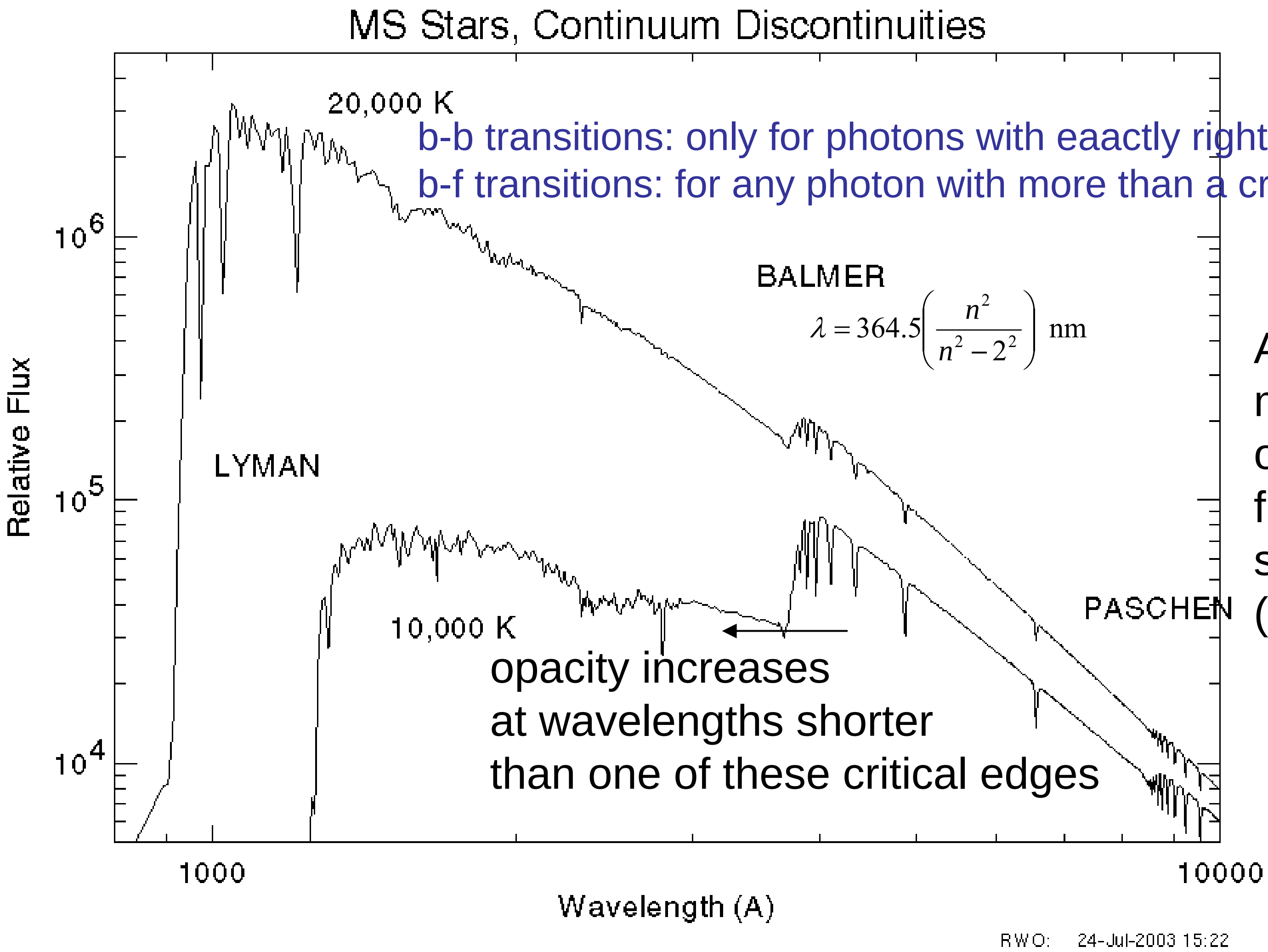
opacity

- Deep in a stellar interior: free-free absorption & electron scattering are more important
- Outer layers of a star: free-bound & bound-bound absorption are more important
- cool stars and sub-stellar objects (brown dwarfs, gas giant planets): molecular or even grain opacities become important

Balmer & Lyman jumps

http://www.astro.virginia.edu/class/oconnell/astr511/lec3-f03.html

absorption discontinuities, ionization edges of abundant ions



b-b transitions: only for photons with eaactly right energy (discrete)

b-f transitions: for any photon with more than a critical energy (continuum)

$$\lambda = 364.5\left(\frac{n^2}{n^2 - 2^2}\right)\ \mathrm{nm}$$

opacity increases
at wavelengths shorter
than one of these critical edges

Application:
measure redshift
of a galaxy by a
filter on the blue
side of the edge
(e.g. Lyman dropout)

$$z = \frac{\Delta\lambda}{\lambda_{\mathrm{rest}}}$$

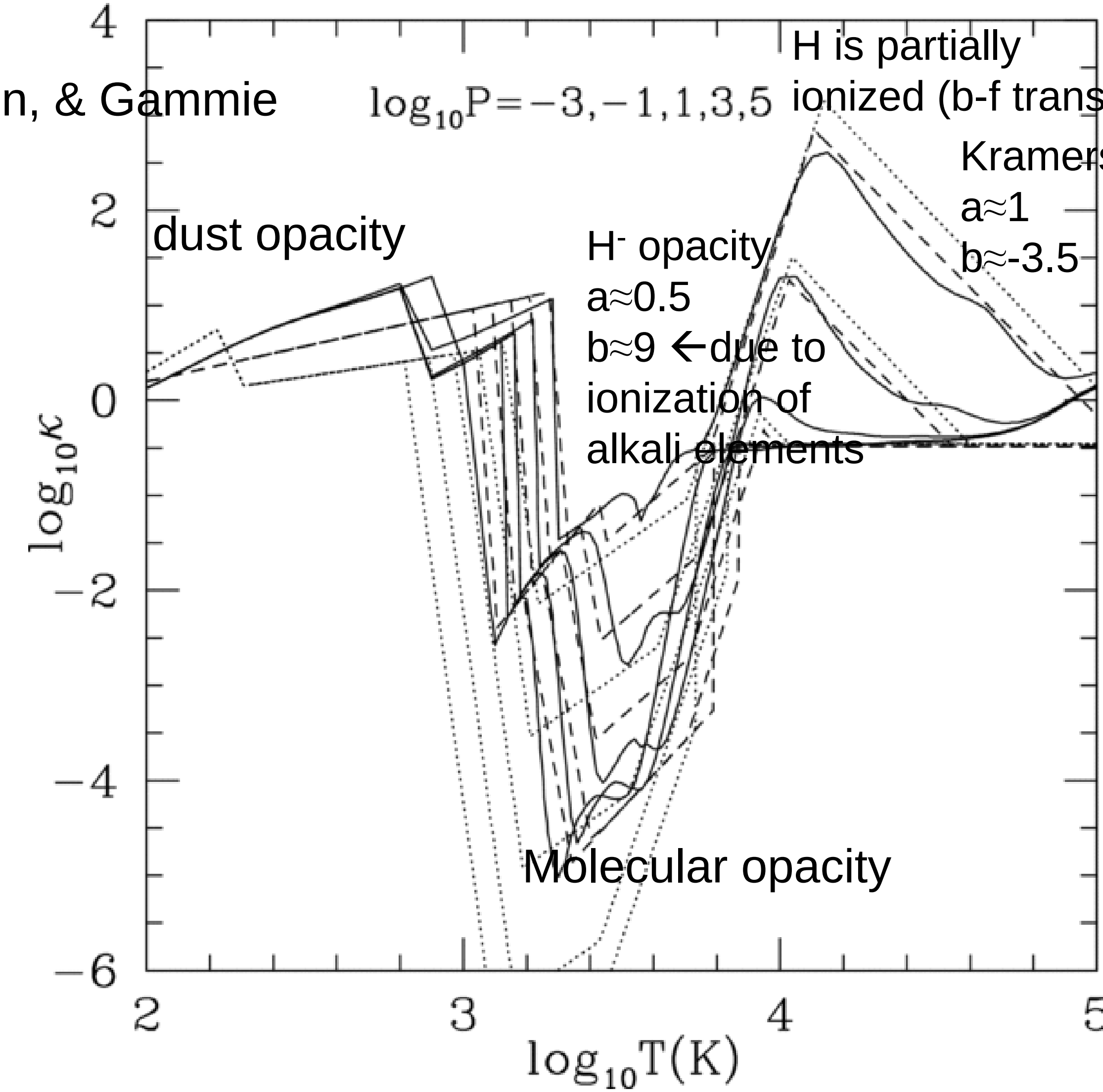
Balmer & Lyman jumps

Note that in the preceding slide, I present an example to demonstrate bound-bound and bound-free opacities in stellar atmospheres, which are not optically thick. Therefore, the opacity should not be expressed by the Rosseland mean (averaged over all possible wavelengths), but is a function of wavelength. Nonetheless, I hope that the example gives you a physical sense of how different sources of opacity works.

Rosseland mean Opacity

tabulated, but can be roughly expressed as $\kappa = \kappa_0(X_i)\rho^a T^b$

Zhu, Hartmann, & Gammie 2009



a=b=0

electron scattering (a flat curve)

summary

- mean molecular weight: average molecular weight per free particle in a mixture of ideal gases
- Saha eqn: determine thermal ionization fraction, which in tern affects thermodynamic quantities.
- opacity sources: electron scattering, f-f, b-f, b-b, H^-, molecular, grain; Rosseland mean opacities are tabulated but can be approximately expressed in terms of a power law of density/pressure and temperature in different regimes.