Hölmström (1979)

- This paper proposes a model which has later become the standard formulation for the principal-agent relation.
- Two contributions:
  (1) Making clear the tradeoff between risk and incentives in the moral hazard problem.
  (2) Investigating the informational value of signals.

Model

- A principal (P) hires an agent (A) to engage in a production.
- Output y, which depends on A's effort level, is observable and verifiable.
- The level of A's effort, e, is his private information.
- Let $f(y;e)$ be the density function of y, conditional on e.
- $f(y;e)$ first-order stochastically dominates $f(y;e')$ for all $e' < e$. That is, $F(y;e) \leq F(y;e')$ for all $e > e'$, with strict inequality holds for some y.

Model

- Since e is unobservable and y is observable and correlated with e, the way to provide incentives to A is to offer A a contract which is function of $y: w(y)$.
- Utility of A:
$$u(w) - v(e);$$
where w is wage and e is effort level. Assume $u' > 0$, $u'' \leq 0$, $v' > 0$, $v'' > 0$.
- Utility of P:
$$U(y - w);$$
where $U' > 0$, $U'' \leq 0$.
- P offers a take-it-or-leave-it contract to A, who decides whether to accept or not.
- Reservation utility of A: $\underline{u}$.

First-Best Case

- In order to emphasize the importance of the observability of effort, let's for the moment assume that e is observable.
- The principal's problem is easy:

  (i) If he intends to implement any effort level $\bar{e}$, the contract is

$$w = \begin{cases} w(y); & \text{if } e = \bar{e}, \\ 0; & \text{if } e \neq \bar{e}; \end{cases} \tag{1}$$

  where $w(y)$ is such that

$$\int_y u(w(y))f(y;\bar{e})dy - v(\bar{e}) = \underline{u}.$$

  (ii) The value of $\bar{e}$ is then chosen to maximize the principal's utility:

$$\bar{e} \in \arg\max_e \int_y U\big(y - w(y)\big)f(y;e)dy.$$

First-Best Case

- Note that, even if e is observable, w can't depend only on e. Because of the need to share risk, it also depends $w(y)$. Reason:

  (a) if w depends only on e, at optimum A's wage is fixed;

  (b) P then bears all the risk from production;

  (b) This is not optimal because there is room for risk-sharing, as both A and P are risk-averse.

First-Best Case

- These two steps can be combined into one optimization program:

$$\begin{aligned} \max_{e,w(y)} \quad & \int_y U\big(y - w(y)\big) f(y;e)dy \\ \text{s.t.} \quad & \int_y u(w(y)) f(y;e)dy - v(e) \geq \underline{u}. \end{aligned}$$

- The constraint above is called the individually rational constraint.

First-Best Case

- The FOC of the solution is

$$\frac{U'\big(y - w(y)\big)}{u'\big(w(y)\big)} = \lambda \quad \forall y; \tag{2}$$

where $\lambda \geq 0$ is a constant.

- Equation (1) implies that

$$\frac{U'\big(y - w(y)\big)}{U'\big(y' - w(y')\big)} = \frac{u'\big(w(y)\big)}{u'\big(w(y')\big)} \quad \forall y,\, y'.$$

First-Best Case

- From (2) it is easy to show that $w'(y) > 0$ and $\big(y - w(y)\big)' > 0$: Both P and A benefit from increases in output.
- In the special case when P is risk neutral and A is risk averse, $w(y)$ is a constant.
- Similarly, when P is risk averse and A is risk neutral, $y - w(y)$ is a constant.
- Whoever is risk-neutral receives a fixed payment, if the other party is risk neutral.
- When e is observable, the needs to provide incentives and to share risks can be separated.

Formulation when Effort is Unobservable: Second-Best

- Suppose e is unobservable.
- The objective function of P (program M):

$$\max_{e,\,w(y)} \quad \int U\big(y - w(y)\big) f(y;e)dy$$

$$\text{s.t.} \quad \int u\big(w(y)\big) f(y;e)dy - v(e) \geq \underline{u}; \qquad (3;\ \text{IR})$$

$$e \in \arg\max_{e'} \int u\big(w(y)\big) f(y;e')dy - v(e'). \qquad (4;\ \text{IC})$$

Formulation when Effort is Unobservable: Second-Best

- Constraint (3) is the individually rational (IR) constraint:
  In order for A to accept contract $w(y)$, the expected utility from $w(y)$ must be at least the reservation value.
- Constraint (4) is the incentive compatibility (IC) constraint:
  Since e is not observable, A certainly chooses the level of e to maximize his expected utility, given the incentive structure $w(y)$.
- Note that although e is not observable to P, he can actually infer its value by solving for (4).
- Compared to the case when e is observable, now an additional constraint (IC) must be satisfied.

First-Order Condition Approach

- Replace (4) by its first-order condition:

$$\int u\big(w(y)\big)f_e(y;e)dy - v'(e) = 0. \tag{4'}$$

- The Lagrarangian of program M is

$$\begin{aligned} L = & \int U\big(y - w(y)\big)f(y;e)dy \\ & + \lambda\Big[\int u\big(w(y)\big)f(y;e)dy - v(e) - \underline{u}\Big] \\ & + \mu\Big[\int u\big(w(y)\big)f_e(y;e)dy - v'(e)\Big]; \end{aligned}$$

where $\lambda \geq 0$ and $\mu \geq 0$ are constants.

First-Order Condition Approach

- FOC:

$$\frac{U'\big(y - w(y)\big)}{u'\big(w(y)\big)} = \lambda + \mu \frac{f_e(y;e)}{f(y;e)} \qquad \forall y,$$

$$\int U\big(y - w(y)\big) f_e(y;e)dy$$
$$+ \mu\Big[\int u\big(w(y)\big) f_{ee}(y;e)dy - v''(e)\Big] = 0. \qquad (5)$$

First-Order Condition Approach

- **Proposition 1:** Compared to the case when e is observable, there is efficiency loss in the optimal contract.

- Proof:

  Suffice to show that IC constraint is binding, i.e., $\mu > 0$. If, on the contrary, $\mu \leq 0$. Then for all y with $f_e(y;e) \geq 0$,

$$\frac{U'\big(y - w(y)\big)}{u'\big(w(y)\big)} = \lambda + \mu \frac{f_e(y;e)}{f(y;e)} \leq \lambda = \frac{U'\big(y - w_\lambda(y)\big)}{u'\big(w_\lambda(y)\big)};$$

  where $w_\lambda(y)$ is the solution of (2) when the lagrange multiplier for (2) is replaced by the value of λ in program M.

- As $U'' \leq 0$ and $u'' \leq 0$, we know that $y - w(y) \geq y - w_\lambda(y)$.

First-Order Condition Approach

- On the other hand, if y is such that $f_e(y;e) \leq 0$, then we can similarly show that $y - w(y) \leq y - w_\lambda(y)$.
- Combine the above two results we know that

$$\begin{aligned}\int U\big(y - w(y)\big) f_e(y;e)dy &\geq \int U\big(y - w_\lambda(y)\big) f_e(y;e)dy \\ &= U\big(y - w_\lambda(y)\big) F_e(y;e)\Big|_0^\infty \\ &- \int U'\big(y - w_\lambda(y)\big)\big(y - w_\lambda(y)\big)' F_e(y;e)dy > 0, \qquad (6)\end{aligned}$$

where the last inequality comes from the facts that $F_e(y;e) \leq 0$ and that $(y - w_\lambda(y))' > 0$. However, we know that the second term in (5) is positive by the facts that $\mu \leq 0$ and SOC. Thus (5) contradicts with (6). QED

Is Wage Increasing in Output?

- Differentiate the first equation of (5) with respect to y:

$$w'(y) = \frac{U'' - \mu \frac{\partial (f_e/f)}{\partial y} u'}{U'' + U' \frac{u''}{u'}}.$$

- $w'(y)$ can be negative when $\partial (f_e|f)/\partial y$ is sufficiently negative: A might be paid less when output increases.

- We can guarantee $w'(y) > 0$ only if $\frac{\partial (f_e/f)}{\partial y} > 0$.

Is Wage Increasing in Output?

- In general, $w(y)$ is not always increasing in y:

  Example: Let $\tilde{y} = \tilde{\theta} + e$, when $e \in [0, 1/2]$. $\tilde{\theta} = 1$ or 0 with equal probability.
  The output level $y = 1/2$ must be a result of e being 1;
  and the output level $y = 1$ must be a result of e being 0.
  P infers lower effort when he observes higher output.

- Lesson: Sometimes P infers a lower effort level when he observes higher output. In that case it is natural that A's wage is lower when output is higher.

Is Wage Increasing in Output?

- Needed in order for $w'(y) > 0$: $f(y;e)/f(y;e')$ increases in y for all $e > e'$. That is,

$$\frac{f(y;e+\Delta e)}{\Delta e f(y;e)} \text{ increases in } y.$$

  Make $\Delta e \to 0$, we have $f_e(y;e)/f(y;e)$ increases in y.

- The value of f_e/f is called "likelihood ratio".
- The property that f_e/f is increasing in y is called "monotone likelihood ratio property" (MLRP).
- Wage is guaranteed to increase with output only when the density function of output satisfies MLRP.

The Value of Information

- Suppose in addition to y, there is also another verifiable signal x.
- Should the optimal contract be a function of x also?
  Not necessarily.
- FOC:
  $$\frac{U'\big(y - w(x,y)\big)}{u'\big(w(x,y)\big)} = \lambda + \mu \frac{f_e(x,y;e)}{f(x,y;e)};$$
  where $f(x,y;e)$ is joint density of x and y.
- If $f_e/f = h(y,e)$, which is a function of y only, then $w(x,y)$ will be independent of x, meaning that the optimal contract needs not incorporate the information offered by the signal x.

The Value of Information

- This actually means that $f(x, y; e)/f(x, y; e')$ is independent of x: Changes in x does not change the relative probability between any two e and e'.
- Note that if
$$\frac{f_e(x, y, e)}{f(x, y, e)} = h(y, e),$$
then integrating both sides upon e we have
$$\log f(x, y, e) = \bar{g}(x, y) + \bar{H}(y, e);$$
i.e.,
$$f(x, y, e) = g(x, y)H(y, e). \tag{7}$$

The Value of Information

- Equation (7) exactly means that y is a sufficient statistics for (x, y), i.e., $f(y|x;e) = f(y;e)$. (The Factorization Theorem.)

- In this case

$$\frac{f_e(x,y;e)}{f(x,y;e)} = \frac{h(y,e)f(x,y;e)}{f(x,y;e)} = h(y,e);$$

which is independent of x.

- **Proposition:** An additional signal x is of informational value (and thus should be written into the contract) if and only if y is not a sufficient statistic of x, or, equivalently, $f(\cdot)$ cannot be factorized in a way that $f(x,y,e) = g(x,y)H(y,e)$ for some $g(\cdot)$ and $H(\cdot)$.