Competitive Equilibrium (from Varian's Textbook)

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Introduction

- The existence and welfare theorems of the competitive market is a crown jewel of the achievenments of the economic science in the 20th century.
- It lays the foundation for the superiority of the *laissez-faire* economy or, the capitalism.
- This theory essentially shows that, if the market is perfectly competitive, then there exists a price system in which
  - (i) every consumer maximizes his utility;
  - (ii) every producer maximizes its profit;
  - (iii) total demand equal total supply for every good; and
  - (iv) under this price system, the economy attains Pareto optimal.

Start with the Exchange Economy

- n consumers $(i = 1, ...n)$, m goods $(j = 1, ...m)$.
- Price of good $j : p_j$.
- $\mathbf{x}_i \equiv (x_{i1}, x_{i2}, ..., x_{im})$: consumer $i's$ consumption.
- $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n)$: an allocation.
- $\mathbf{p} = (p_1, ..., p_m)$= price vector, $p_j \geq 0$, $\forall j$.

Start with the Exchange Economy

- $\mathbf{e}_i = (e_{i1}, ..., e_{im})$: endowment of consumer i.
- $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n)$
- Let

$$\mathbf{x}_i(\mathbf{p}, \mathbf{pe}_i) = argmax_{\mathbf{x}}\ u_i(\mathbf{x}).$$

$$s.t.\ \mathbf{px}_i = \mathbf{pe}_i.$$

- $\mathbf{x}_i(\mathbf{p}, \mathbf{pe_i})$ is the demand function of consumer i.

Walrasian Equilibrium

- A price system and an allocation, $(\mathbf{p}^*, \mathbf{x}^*)$ is a Walrasian equilibrium (WE) if

$$\sum_{i=1}^{n} \boldsymbol{x}_{\mathbf{i}}(\mathbf{p}, \mathbf{pe}_{\mathbf{i}}) \leq \sum_{\mathbf{i}} \mathbf{e}_{\mathbf{i}}$$

- The inequality because a good might be free, and consumer might be satiated, so that quantity demanded is finite (and less than total endowment) even if price is zero.
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$$\boldsymbol{z}(\mathbf{p}) \equiv \sum_{i=1}^{n} [\mathbf{x}_i(\mathbf{p}, \mathbf{pe}_i) - \mathbf{e}_i];$$

which is the excess demand function.

Walrasian Equilibrium

Walras Law: *For any price* $\mathbf{p}$, $\mathbf{p}\boldsymbol{z}(\mathbf{p}) = 0$.

Proof.

$\mathbf{pz}(\mathbf{p}) = \sum_{i=1}^{n} \mathbf{px}_i(\mathbf{p}, \mathbf{pe}_i) - \sum_{i=1}^{n} \mathbf{pe}_i = \sum_{i=1}^{n} [\mathbf{px}_i(\mathbf{p}, \mathbf{pe}_i) - \mathbf{pe}_i] = 0$ □

Walras law is actually just a budget constraint requirement.

If $\mathbf{p}^*$ *is the price corresponds to the Walrasian equilibrium and* $z_j(\mathbf{p}^*) < 0$, *then it must be that* $p_j^* = 0$.

Proof.

Since in WE, $\mathbf{z}(\mathbf{p}^*) \leq 0$, *and, since* $p_j^* \geq 0 \, \forall j, p_j^* z_j(p^*) \leq 0 \, \forall j$. *If* $z_j(\mathbf{p}^*) < 0$ *and* $p_j^* > 0$, *it must be that* $\mathbf{p}^* \mathbf{z}(\mathbf{p}^*) < 0$, *contradicting Walras Law.* □

Walrasian Equilibrium

- **A1**: If $p_j = 0$, then $z_j(\mathbf{p}) > 0$.
- **A1** is an assumption of aggregate non-satiation.
- Note that since $\mathbf{x}_i(\lambda\mathbf{p}, \lambda\mathbf{pe}_i) = \mathbf{x}_i(\mathbf{p}, \mathbf{pe}_i)$ for all $\lambda > 0, \mathbf{x}(\lambda\mathbf{p}, \lambda\mathbf{pe}_i) = \mathbf{x}(\mathbf{p}, \mathbf{pe}_i)$.
- In this case, we can restrict $\mathbf{p}$ to belong to the $(m-1)$-dimensional simplex $S^{m-1} = \{\mathbf{p} \in \Re^m_+ | \sum_{j=1}^m p_j = 1\}$.

Existence of WE

Theorem

If $\mathbf{z}(\cdot)$ *is continuous on* S^{m-1} *and that* $\boldsymbol{pz}(p) \equiv 0$*, then there exists* $p^* \in S^{m-1}$ *such that* $\mathbf{z}(\mathbf{p}^*) \leq 0$*.*

Proof.

Define $\mathbf{g} : S^{m-1} \to S^{m-1}$ *by*

$$g_j(p) = \frac{p_j + max(0, z_j(\mathbf{p}))}{1 + \sum_{l=1}^{k} max(0, z_l(\mathbf{p}))}, j = 1, ..., m.$$

$g_j(\cdot)$ *is continuous because* $z_j(\cdot)$ *is. Also* $\mathbf{g}(p) \in S^{m-1}$ *because* $\sum_{j=1}^{m} g_j(p) = 1$*. By Brouwer's fixed-point theorem, there exists* $p^* \in S^{m-1}$ *such that* $\mathbf{p}^* = \mathbf{g}(\mathbf{p}^*)$*.*

Existence of WE

Proof.(Cont.)

That is,

$$p_j^* = \frac{p_j^* + max(0, z_j(\mathbf{p}^*))}{1 + \sum_l max(0, z_l(\mathbf{p}^*))} \text{ for } j = 1, ..., k.$$

This implies

$$p_i^* \sum_{l=1}^{k} max(0, z_l(\mathbf{p}^*)) = max(0, z_l(\mathbf{p}^*)), \ j = 1, ..., k.$$

Multiply both sides by $z_j(\mathbf{p})$:

$$z_j(\mathbf{p}^*) p_j^* \left[\sum_{l=1}^{k} max(0, z_l(\mathbf{p}^*)) \right] = z_j(\mathbf{p}^*) max(0, z_j(\mathbf{p}^*)), \ j = 1, ..., k.$$

Existence of WE

Proof (Cont.)

Summing up:

$$\left[\sum_{l=1}^{k} max(0, z_l(\mathbf{p}^*))\right] \sum_{j=1}^{k} p_j^* z_j(\mathbf{p}^*) = \sum_{j=1}^{k} z_j(\mathbf{p}^*) max(0, z_j(\mathbf{p}^*)).$$

By Walras Law,

$$\sum_{j=1}^{k} z_j(\mathbf{p}^*) max(0, z_j(\mathbf{p})) = 0.$$

Therefore

$$z_j(\mathbf{p}^*) \leq 0. \ \forall j = 1, ..., m.$$

$\square$

Existence of WE

- If good j is such that $z_j(\mathbf{p}^*) < 0$, then we know that $p_j^* = 0$. However, **A1** requires that $z_j(\mathbf{p}^*) > 0$ for any $p_j^* > 0$. Therefore, if we impose **A1**, $z(\mathbf{p}^*) = 0$.

First Welfare Theorem

- An allocation $\mathbf{x}$ is called feasible if $\sum_{i=1}^{n} \mathbf{x}_i = \sum_{i=1}^{n} \mathbf{e}_i$. It is Pareto efficient (PE) if there does not exist another feasible allocation $\mathbf{x}'$ so that every agent i prefers $\mathbf{x}'$ to $\mathbf{x}$.

First Welfare Theorem *Assume* **A1**, *holds. If* $(\mathbf{x}, \mathbf{p})$ *is WE, then it is PE.*

Proof.

First note that since every agent i maximizes utility given endowment in a WE, it is equivalent to say that if there is any $\mathbf{x}'_i$ preferred by i to $\mathbf{x}_i$, then it must be $p\mathbf{x}_i < p\mathbf{x}'_i$.

First Welfare Theorem

Suppose $(\mathbf{x}, \mathbf{p})$ is WE but not PE, then there exist a feasible $\mathbf{x}'$ so that every i prefer $\mathbf{x}'_i$ to $\mathbf{x}_i$. Therefore, $\mathbf{p}\mathbf{x}_i < \mathbf{p}\mathbf{x}'_i$ for all i. This implies

$$\mathbf{p}\sum_{i=1}^{n}\mathbf{e}_i = \mathbf{p}\sum_{i=1}^{n}\mathbf{x}'_i > \mathbf{p}\sum_{i=1}^{n}\mathbf{x}_i,$$

a contradiction.

Second Welfare Theorem

- Let $\mathbf{x}^*$ be a PE allocation such that $\mathbf{x}_i^* > 0$ for all i. Assume preferences are convex, continuous, and monotonic. Then $\mathbf{x}^*$ is a WE with endowment $\mathbf{e} = \mathbf{x}^*$.

Proof.

Let $P_i = \{\mathbf{x}_i | \mathbf{x}_i \succ_i \mathbf{x}_i^*\}$.

Define $P = \sum_{i=1}^n P_i = \{\mathbf{z} | \mathbf{z} = \sum_{i=1}^n \mathbf{x}_i, \mathbf{x}_i \in P_i\}$.

P is convex since every P_i is.

Let $\bar{\mathbf{e}} = \sum_{i=1}^n \mathbf{x}_i^*$. Obviously, $\bar{\mathbf{e}} \notin P$. Therefore, by the seperating hyperplane theorem, there exists $\mathbf{p} \neq 0$ such that

$$\mathbf{p}\mathbf{z} \geq \mathbf{p}\bar{\mathbf{e}} \, \textit{for all} \, \mathbf{z} \in P,$$

Second Welfare Theorem

Proof.(Cont.)

i.e.,

$$\mathbf{p}(\mathbf{z} - \sum \mathbf{x}_i^*) \geq 0 \ \ \forall \mathbf{z} \in P.$$

Want to show that $\mathbf{p}$ is a WE price vector. First, $\mathbf{p} \geq 0$: Consider $\bar{\boldsymbol{e}} + \mathbf{v}_i$, where $\mathbf{v}_i = (0, ..., 1, 0, ...0)$. Obviously it lies in P. Therefore

$$\mathbf{p}(\bar{\mathbf{e}} + \mathbf{v}_i - \bar{\mathbf{e}}) \geq 0,$$

which reduces to $p_i \geq 0$.

Second Welfare Theorem

Proof.(Cont.)

Second, we want to show that if $\mathbf{x}_i \succ_i \mathbf{x}_i^*$, then $\mathbf{p}\mathbf{x}_i \geq \mathbf{p}\mathbf{x}_i^*$ for all i:
Constuct an allocation $\mathbf{z}$ as follows

$$\mathbf{z}_i = (1-\theta)\mathbf{x}_i,$$
$$\mathbf{z}_j = \mathbf{x}_j^* + \frac{\theta}{n-1}\mathbf{x}_i.$$

If θ is small enough, then i still prefers $\mathbf{z}_i$ to $\mathbf{x}_i^*$ (by continuity). Moreover, j prefers $\mathbf{z}_j$ to $\mathbf{x}_j^*$ for all j (by monotonicity). Therefore, $\sum_{i=1}^n \mathbf{z}_i \in P$.

Second Welfare Theorem

Proof.(Cont.)

Again, by separating hyperplane theorem,

$$\mathbf{p}\sum \mathbf{z}_i \geq \mathbf{p}\sum_{i=1}^{n} \mathbf{x}_i^*.$$

That is,

$$\mathbf{p}\left[\mathbf{x}_i(1-\theta) + \sum_{j\neq i} \mathbf{x}_j^* + \mathbf{x}_i\theta\right] \geq \mathbf{p}\left[\mathbf{x}_i^* + \sum_{j\neq i} \mathbf{x}_j^*\right], \text{implying}$$

$$\mathbf{p}\mathbf{x}_i \geq \mathbf{p}\mathbf{x}_i^*.$$

Second Welfare Theorem

Proof (Cont.)

Finally, we show $\mathbf{x}_i \succ_i \mathbf{x}_i^*$ implies $\mathbf{p}\mathbf{x}_i > \mathbf{p}\mathbf{x}_i^*$:

We already know that $\mathbf{p}\mathbf{x}_i \geq \mathbf{p}\mathbf{x}_i^*$.

Suffice to show that $\mathbf{p}\mathbf{x}_i = \mathbf{p}\mathbf{x}_i^*$ leads to contradiction.

By continuity we know that for $\theta \in (0,1), \theta\mathbf{x}_i \succ_i \mathbf{x}_i^*$ if θ is sufficiently close to 1, so that $\theta\mathbf{p}\mathbf{x}_i \geq \mathbf{p}\mathbf{x}_i^*$.

However, $\mathbf{p}\mathbf{x}_i = \mathbf{p}\mathbf{x}_i^*$ implies that $\theta\mathbf{p}\mathbf{x}_i < \mathbf{p}\mathbf{x}_i^*$, a contradiction to above. $\square$