Section 7

Isoperimetric Problem

An optimization problem may be subject to an integral constraint:

$$\max \int_{t_0}^{t_1} F(t, x, x')\, dt \tag{1}$$

$$\text{subject to} \quad \int_{t_0}^{t_1} G(t, x, x') dt = B, \quad x(t_0) = x_0, \quad x(t_1) = x_1, \tag{2}$$

where F and G are twice continuously differentiable functions and B is a given number. For example, the problem of maximizing the area enclosed by a straight line and a string of length B can be posed in this form. Let the straight line extend from $(t_0, x_0) = (0, 0)$ to $(t_1, x_1) = (t_1, 0)$. Then the area under the curve will be given by (1) with $F(t, x, x') = x$. The constraint on string length is given by (2) with $G(t, x, x') = [1 + (x')^2]^{1/2}$. (Recall Example 1.5.) In this problem, the perimeter is constant, specified by (2)—hence the name "isoperimetric." Such an example has provided the name for the whole class of problems given by (1) and (2). Another example of the form of (1) and (2) was given in Exercises 5.5 and 5.6,

$$\max \int_0^T e^{-rt} P(x)\, dt \tag{3}$$

$$\text{subject to} \quad \int_0^T x\, dt = B, \tag{4}$$

where $x(t)$ is the rate of extraction of a resource, B the initial endowment of the resource, and $P(x)$ the profit rate at t if the resource is extracted and sold at rate $x(t)$. Because of the special structure of (4), one can convert the isoperimetric constraint (4) into a fixed endpoint constraint, by defining

$$y(t) = \int_0^t x(s)\,ds \tag{5}$$

as the amount of resource extracted by time t. Then $y'(t) = x(t)$ and (3) and (4) are equivalently stated as

$$\max \quad \int_0^T e^{-rt} P(y')\,dt \tag{6}$$

$$\text{subject to} \qquad y(0) = 0, \quad y(T) = B. \tag{7}$$

Typically, there is no simple transformation to eliminate an isoperimetric constraint. However, recall that in a constrained calculus optimization problem, one may either use the constraint to eliminate a variable (yielding an equivalent unconstrained problem), or the constraint may be appended to the objective with a Lagrange multiplier and equivalent necessary conditions developed (see Section A5). A Lagrange multiplier technique works here also. For instance, appending (4) to (3) with a Lagrange multiplier gives

$$\begin{aligned} L &= \int_0^T e^{-rt} P(x)\,dt - \lambda\left(\int_0^T x\,dt - B\right) \\ &= \int_0^T \left[e^{-rt} P(x) - \lambda x\right] dt + \lambda B. \end{aligned} \tag{8}$$

A necessary condition for x to maximize the augmented integrand (8) is that it satisfy the Euler equation

$$e^{-rt} P'(x) = \lambda. \tag{9}$$

In agreement with the findings of Exercise 5.6, the present value of marginal profits is constant over the planning period.

In the general case, (1) and (2), we append constraint (2) to (1) by an undetermined multipler λ. Any admissible function x satisfies (2), so for such an x,

$$\int_{t_0}^{t_1} F(t,x,x')\,dt = \int_{t_0}^{t_1} [F(t,x,x') - \lambda G(t,x,x')]\,dt + \lambda B. \tag{10}$$

The integral on the left attains its extreme values with respect to x just where the integral on the right does; λ then is chosen so that (2) is satisfied. The Euler equation for the integral on the right is

$$F_x - \lambda G_x = d(F_{x'} - \lambda G_{x'})/dt. \tag{11}$$

From (A5.11) the Lagrange multiplier method rests on the supposition that the optimal point is not a stationary point of the constraining relation; this prevents division by zero in the proof. An analogous proviso pertains here for a similar reason. Thus, a necessary condition for solution to (1) and (2) may be stated as follows:

If the function x^* is an optimal solution to (1) *and* (2) *and if x^* is not an extremal for the constraining integral* (2), *then there is a number λ such that $x^*(t)$, λ satisfy* (2) *and* (11).

Example 1

$$\min \quad \int_0^1 [x'(t)]^2\, dt$$

$$\text{subject to} \qquad \int_0^1 x(t)\, dt = B, \quad x(0) = 0, \quad x(1) = 2.$$

The augmented integrand is $(x')^2 - \lambda x$. Its Euler equation $\lambda + 2x'' = 0$ has the solution

$$x(t) = -\lambda t^2/4 + c_1 t + c_2.$$

Three constants are to be determined—λ, c_1, c_2—using the integral constraint and boundary conditions:

$$\int_0^1 x\, dt = \int_0^1 (-\lambda t^2/4 + c_1 t + c_2)\, dt = B,$$

$$x(0) = c_2 = 0, \qquad x(1) = -\lambda/4 + c_1 + c_2 = 2.$$

Hence

$$c_1 = 6B - 4, \qquad c_2 = 0, \qquad \lambda = 24(B - 1).$$

Example 2. For

$$\max \quad \int_0^T x\, dt$$

$$\text{subject to} \qquad \int_0^T [1 + (x')^2]^{1/2}\, dt = B, \quad x(0) = 0, \quad x(T) = 0,$$

the augmented integrand $x - \lambda[1 + (x')^2]^{1/2}$ has Euler equation

$$1 = -d\left(\lambda x'/\left[1 + (x')^2\right]^{1/2}\right)/dt.$$

Separate the variables and integrate:

$$t = -\lambda x'/\left[1 + (x')^2\right]^{1/2} + k.$$

Solve for x' algebraically:

$$x' = (t - k)/\left[\lambda^2 - (t - k)^2\right]^{1/2}.$$