Repeated Games: Theory and Applications

Kong-Pin Chen

The Stage Game

- Let $G = \{\{A_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n\}$ be a normal form game.
- $N = \{1, ..., n\}$: set of players
- A_i: set of pure strategies for player i. $q_i \in A_i$
- $\Delta(A_i)$: set of mixed strategies of i.
- $\pi_i : A_1 \times ... \times A_n \to R$ is payoff function for i.
- $q \equiv (q_1, ...q_n), \pi \equiv (\pi_1, ..., \pi_n)$.
- $A \equiv A_1 \times A_2 \times ... \times A_n$. $A^t \equiv \underbrace{A \times ... \times A}_{t \text{ times}}$.

The Repeated Game

- Let $G^{\infty}(\delta)$ be the infinitely repeated version of G with discount factor $\delta \in (0, 1)$.
- If we replace ∞ by $T < \infty$, it is a finitely repeated game with T periods.
- A pure strategy for player i in G^{∞}, σ_i, is a sequence of function $\sigma_i(1), \sigma_i(2), ..., \sigma_i(t)$..., one for each period t. $\sigma_i(1) \in A_i$.
- For $t = 2, ..., \sigma_i(t) : A^{t-1} \rightarrow A_i$ is a function from all player's past action to A_i
- $\sigma = (\sigma_1, ...\sigma_n)$ is called a strategy profile.
- Let $\sum_i$ be the set of all possible strategies for i.

The Repeated Game

- $\{q(t)\}_{t=1}^{\infty}$, with $q(t) \in A$ for all t, is called a path, denoted Q
- Ω is the set of all paths.
- Every strategy profile σ generates a corresponding path $Q(\sigma) = \{q(\sigma)(t)\}_{t=1}^{\infty}$; where $q(\sigma)(1) = \sigma(1)$, and $q(\sigma)(t) = \sigma(t)(q(\sigma)(q), ..., q(\sigma)(t-1))$.
- For any $Q \in \Omega$, $v_i(Q) \equiv (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1}\pi_i(q(t))$.
- Also denote $v_i(\sigma) \equiv v_i(Q(\sigma))$.

The Repeated Game

- Let $h = (q(1), ...q(t)) \in A^t$ be any t-period history.
- $\sigma|_h$ is the strategy profile induced by σ on the subgame following h.
- σ is a SPE if $\sigma|h$ is a NE of the remaining game for any possible history h.
- $\sigma|_h$ is called a *continuation strategy*, and $v_i(\sigma|_h)$ player i's *continuation payoff*, after history h.
- **A1**:$\{\pi(q)|q \in A\}$ is bounded.

Some Simple Facts of Finitely Repeated Games

- Let $G^T(\sigma)$ be the game formed by repeating G for T times, with $\sigma \in (0, 1]$.
- Can solve for equilibrium by backward induction.
- However, computationally complicated.
- **Fact 1**: If G has a unique NE, then $G^T(\sigma)$ has a unique SPE.
- **Fact 2**: When G has more than one NE, there can be multiple equilibrium.

Some Simple Facts of Finitely Repeated Games

- Example: $T=2$.

	A_2	B_2	C_2
A_1	1,1	0,1	-1,1
B_1	2,1	2,2	0,1
C_1	0,0	1,-1	0,0

Lots of SPE. One of them is $((A_1, A_2), (B_1, B_2))$, if $\delta \geq 1/2$.

Infinitely Repeated Games: One-Shot Deviation Principle

- When $\mathbf{A_1}$ holds, in order to check for SPE, we only have to check for "one-shot" deviation.
- One shot deviation of a player means that the player deviates once, then confirms to the strategy thereafter.
- Let σ be a strategy profile. If no player can profitably deviate from σ by any "one-shot" deviation after any possible history, then σ is a SPE.

One-Shot Deviation Principle

Proof.

If one-shot deviation is not profitable, then finite-shot will not be profitable either. It remains to show that infinitely-shot deviation is also unprofitable. Suppose, on the contrary, an infinite deviation increases profit by d. By **A1**, there exist an T so that the deviator's continuation payoff after T is smaller than $d/2$. This means this deviation must already has gained a profit (by deviating) before period T, a contradiction. QED □

Folk Theorem

- When a normal form game G is infinitely repeated, the SPE set of $G^\infty(\delta)$ will generally "explode", even if G has a unique NE.
- Example: Prisoner's Dilemma

	A_2	B_2
A_1	2,2	-1,3
B_1	3,-1	0,0

- It has only one NE, (B_1, B_2). In particular, the efficient outcome (A_1, A_2) cannot be supported as NE.

Folk Theorem

- Suppose it is infinitely repeated. Consider the following strategy: $\sigma_i(1) = A_i$ for $i = 1, 2$. For $t \geq 2, \sigma_i(t) = A_i$ if $q_j(s) = A_j$ for all $s < t$ and for all $j = 1, 2$. Otherwise $\sigma_i(t) = B_i$.
- In words, a player plays A_i if (A_1, A_2) had been played in all previous periods. Once a player i does not play A_i, they switch to (B_1, B_2) forever.
- This is called a "trigger strategy".

Folk Theorem

- This strategy is a SPE if $\delta \geq 1/3$.
- Player can play a non-Nash action profile *in every stage* as a SPE even if the underlying normal form game has only one NE.
- In fact, any (v_1, v_2) with $v_i > 0$ can be supported as a SPE payoff in a similar way. This is one version of the "Folk Theorem".

Folk Theorem

- F be the smallest convex set containing $\{\pi(q)|q \in A\}$. F is the *feasible payoffs set*.
- Let $\underline{v}_i \equiv min_{q_{-i} \in \pi_{j \neq i} \Delta(A_j)} max_{q_i \in \Delta(A_i)} \pi_i(q_i, q_{-i})$.
- $\underline{v}_i$ is called the *minimax payoff for i*.
- Let $F^* = \{v | v \in F, v_i > \underline{v}_i \text{ for all } i\}$.
- Let $\Delta^i(q)$ be the action profile so that $\pi(\Delta^i(q)) = \underline{v}_i$.
- F^* is called the *feasible and individually rational payoffs set*.

Folk Theorem

- Folk Theorem: Suppose the dimension of F^* is n, and that the interior of $F^* \neq \phi$. Then for any $v \in F^*$ there exists $\underline{\delta} \in (0,1)$ so that v is a SPE payoff of $G^\infty(\delta)$ for any $\delta \geq \underline{\delta}$.

Sketch of Proof

The idea is to minimax a player after he deviates, for long enough to wipe out his gain from deviation. To motivate the otehr players to go through this process, they are rewarded by an small amount $\varepsilon > 0$:

Folk Theorem

Sketch of Proof (cont.)

Let $v' \in F^*$ so that $v'_i < v_i$ for all i. Also let $\underline{q}_j$ be the action profile so that $\pi(\underline{q}_j) = (v'_1 + \varepsilon, v'_2 + \varepsilon, ..., v'_j, v'_{j+1} + \varepsilon, ..., v'_n + \varepsilon)$. Consider the following strategy: First they play a path whose payoff is exactly v. If any j deviates, then they play $\Delta^j(q)$ for sufficiently long so that j's gain from deviation is eliminated. After that they play $\underline{q}_j$ forever.

Folk Theorem

Sketch of Proof (cont.)

If any play i deviates during the period when j is being minimaxed, or when they are playing $\underline{q}_j$, then i becomes the deviator, and they will go thorugh the above process against i.

This is a SPE because, first, j will not deviate because his gain from deviation will be eliminated during the period when he is being minimaxed, and $v_j' < v_j$.

Folk Theorem

Sketch of Proof (cont.)

Next, any $i \neq j$ will not deviate when j is being minimaxed, for exactly the same reason that his gain will be wiped out, and that $v_i' < v_i' + \varepsilon$. Finally, no player will deviates when they are playing $\underline{q}_j$, for the same reason as above. QED

Folk Theorem: Finitely Repeated Games

- Consider finitely repeated games with no discount.
- Suppose, for every j, there exists a NE for G, q^j, such that $\pi_j(q^j) > min\{\pi_j(q) | a \in \Delta(A),\ a \text{ is NE of } G\}$. Let $v \in F^*$. Then for any $\varepsilon > 0$, there exists T_0 such that, for all $T \geq T_0$, there is a SPE of G^T, σ^T, such that $|\pi_i(\sigma^T) - v_i| < \varepsilon$.

Simple Strategy Profile

- A *simple strategy profile* (SSP) is one for the players to follow the path Q^0. If any player deviates from Q^0, then switch to Q^i. If any player j again deviates from Q^i, then switch to Q^j, and so on.
- Denote the SSP thus defined as $\sigma(Q^0, Q^1, ..., Q^n)$.
- A *one-shot deviation* of player i from $\sigma \in \sum$ is for him to deviate from $Q(\sigma)$ once, and to conform to $Q(\sigma)$ thereafter.

Simple Strategy Profile

- Let $v_j(Q; t+1) = \sum_{s=1}^{\infty} \delta^s \pi_j(q(t+s))$.

Proposition 1.

Assume **A1**. *The SSP* $\sigma(Q^0; Q^1, ..., Q^n)$ *is a SPE iff* $\pi_j(q_j', q_{-j}^i(t)) - \pi_j(q(t)) \leq v_j(Q^i; t+1) - v_j(Q^j)$ *for all* $q_j' \in A_j / q_j^i(t), j$ *and* i, *and* $t = 1, 2, ...$

- Proposition 1 essentially says that, to check for SPE, we only need to check for one-shot deviation.

Optimal Penal Code

- An *optimal penal code* is a vector of strategy profiles $(\underline{\sigma}^1, \underline{\sigma}^2, ..., \underline{\sigma}^n)$, such that $v_i(\underline{\sigma}^i) = min\{v_i(\sigma) | \sigma$ is SPE of $G^\infty(\sigma)\}$.
- Therefore, $\underline{\sigma}^i$ is the SPE which gives player i the lowest payoff among all.
- A simple penal code (SPC) is defined as $SPC(Q^1, ..., Q^n) \equiv$ $(\sigma(Q^1; Q^1, Q^2, \ldots Q^n), \sigma(Q^2; Q^1, Q^2, \ldots Q^n),$ $\ldots \sigma(Q^n; Q^1, Q^2, \ldots Q^n))$.

Optimal Penal Code

- A $SPC(Q^1, ..., Q^n)$ is called an optimal simple penal code (OSPC) if it is also an optimal penal code.
- Assume **A1**, and that A is compact and π is continuous. Then an OSPC exists.
- Let $(\underline{\sigma}^1, \underline{\sigma}^2, ...\underline{\sigma}^n)$ be an OPC, and $\underline{Q}^i \equiv Q(\underline{\sigma}^i), i = 1, ..., n$. Then $(\underline{Q}^1, \underline{Q}^2, ..., \underline{Q}^n)$ defines an OSPC.

Optimal Penal Code

- Q^0 is a SPE path iff $\pi_j(q_j', q_{-j}^0(t)) - \pi_j(q(t)) \leq v_j(Q^0; t+1) - \underline{v}_j$ for all $q_j' \in A_j$, for all j and for all t.
- Suppose an optimal penal code exists. Then Q^0 is a SPE path iff there exists $Q^i \in \Omega$ for all i so that $\sigma(Q^0; Q^1, ..., Q^n)$ is a SPE.

An Example

- Consider the following game

$$\begin{array}{cccc} & L & M & H \\ L & 10,10 & 3,15 & 0,7 \\ M & 15,3 & 7,7 & -4,5 \\ H & 7,0 & 5,-4 & -15,-15 \end{array}$$

- Can envision it as a Cournot game with 3 output levels, with (L, L) the collusive outcome, and (M, M) the Cournot equilibrium.

An Example

- Trigger strategy to support collusion requires that $\sigma \geq 5/8$. Therefore, can't support a collusion when the player has discount factor $\delta = 4/7$.
- Turns out trigger strategy is not the severest punishment.

An Example

- Let $Q^0 = \{(L, L), (L, L), ...\}$,
  $Q^1 = \{(M, H), (L, M), (L, M)...\}$,
  $Q^1 = \{(H, M), (M, L), (M, L)...\}$.
- Q^i yields a payoff zero to player i.
- Trigger strategy cannot support collusion but the SSP $\sigma(Q^0, Q^1, Q^2)$ can.