1. Lynch pin for N-Abelian Gauge thy

K. Uhlenbeck's thm.

$P \supset G$ = compact Lie group
$\downarrow$
M = n-dim. manifold (.... Riem)
$n = 3, 4$

$\{A_n\}_{n=1}^{\infty}$ sequence of connections on P $E \geq 0$

$\int |F_{A_n}|^2 < E \Rightarrow \exists \Lambda \subset \{1, 2, \ldots\}$
$+ \{g_k \in Aut P\}_{k \in \Lambda}$

$\Rightarrow$ such that $\{g_k^* A_k\}_{k \in \Lambda}$

a) $d = 3$ / converges in weakly in L^2_1 to L^2_1-connection
[$L^2_{1,loc}$ if M is non-comp.]

b) $d = 4$ $\exists$ finite set $\Omega \subset M$ (maybe $\emptyset$) s.t. 1) $|\Omega| \leq C_E$
2) $\{g_n^* A_n\}_{n \in \Lambda}$ conv. in weakly in L^2_1-top in $M \setminus \Omega$ to L^2_1-conn in $P' \to M$ (a diff principal bundle)

2: Definitions / review

$Conn(P) =$ Affine space,

- $A = A_0 + a \leftarrow$ section of v.b. $(P \times_G \mathfrak{g}) \otimes T^*M$ ← Lie alg. G.

- Curv. $A = F_A \in (P \times_G \mathfrak{g}) \otimes \Lambda^2 T^*M$

- $|F_A|^2 = -tr_{\mathfrak{g}}(F_A \wedge * F_A)$

$|\nabla A|^2 + A^2(\nabla A) + A^4$

- $Aut\,P$: $F_{g^* A} = g F_A g^{-1}$

$|F_A| = |F_{g^* A}|$

- L^2_1 : $A = A_0 + a$

$$\|a\|^2_{L^2_1} = \int (|\nabla_{A_0} a|^2 + |a|^2)$$

$\nabla_A =$ cov. derivative

$\nabla_A \sigma = \nabla_{A_0}\sigma + [a, \sigma]$

$\sigma \in C^\infty(P \times_G \mathfrak{g})$

$[F_A, \sigma] = d_A \nabla_A \sigma$

• $\|\cdot\|_{L^2_1}$ depends on A_0 but L^2_1-top. on $Conn(P)$ does not.

3. Uhlenbeck's thm foundational analysis input for – Donaldson inv. $d=4$

– Floer homol. $d=3$

– Thms in $d=2$ relating gauge thy to Riemann surface thy

a) Number of timely reasons to consider gauge thy (Floer homol. Donaldson inv.)

⇒ when G is non compact.

b) ⇒ Lack of "Uhlenbeck" thm is central obstruction. (may be more)

b) What follows is extension of Uhl.'s thm in case $d=3$, $G=PSL(2;\mathbb{C})$.

c) background: $\mathfrak{g} = su(2) \oplus i\,su(2)$

$$\begin{array}{ccc} \mathbb{P} & = & P \times_{SO(3)} PSL(2,\mathbb{C}) \\ \downarrow & & \downarrow \\ M & & M \end{array}$$

d) $\mathrm{Conn}(\mathbb{P}) = \mathrm{Conn}(P) \times C^\infty((P \times_{SO(3)} su(2)) \otimes T^*)$

$\mathbb{A} = \{(A, \alpha)\}$

$F_{\mathbb{A}} = F_A - \alpha \wedge \alpha + d_A \alpha$

$\mathcal{E}(A, a) = \int_M |F_A - \alpha \wedge \alpha|^2 + \int_M |d_A a|^2$

Only $\mathrm{Aut}\,P$ inv, not $\mathrm{Aut}(\mathbb{P})$.

$\mathcal{E}(A, \alpha) = \int_M |F_A - \alpha \wedge \alpha|^2 + \int_M |d_A \alpha|^2 + |d_A * \alpha|^2$

~~Thm~~ $\mathcal{E}(\mathbb{A}) = \inf_{h \in \mathrm{Aut}(\mathbb{P})} \mathcal{E}(h^* \mathbb{A})$

e) L^2_1 - top: $\mathbb{A} = A_0 + a_A + \alpha$

$\int \left(|\nabla_{A_0} a_A|^2 + |a_A|^2 + |\nabla_{A_0} \alpha|^2 + |\alpha|^2 \right)$

f) real line bundle over open set U

$\longleftrightarrow U = \tilde{U} \times_{\mathbb{Z}_2} \mathbb{R}$

2-1 cover of U

d) $Z \subset M$ is Lipschitz curve
when each pt in Z ...... coordinate
nbhood (x_1, x_2, x_3)
such that

$$Z = (\gamma_1(t), \gamma_2(t), t)$$

$$\gamma = (\gamma_2(t), \gamma_3(t)) : (\varepsilon, -\varepsilon) \to \mathbb{R}^2$$

$$\sup_{t,t'} |\gamma(t) - \gamma(t')| \leq c|t-t'|$$

has locally finite length

e) ...... valued ...... section of ......

- dv, $d{*}v$ ...... makes sense
  $W^{1,2}$ ...... H
- Harmonic : $dv = 0$, $d{*}v = 0$

f) exp. $1/2$, Holder cont

$$|f(x) - f(y)| \leq c|x-y|^{1/2}$$

<u>Thm:</u> $\{\mathbb{A}_n\}_{n=1,2,\dots} \in \mathrm{Conn}(\mathbb{P})$

s.t. $E(\mathbb{A}_n) \le E$.

(1) $\exists$ subs. $\Lambda \subset \{1,2,\dots\}$ s.t. plus

$\{h_n\}_{n\in\Lambda}$ s.t. $\{h_n^*\mathbb{A}_n = A_0 + \nu\mathfrak{a}_n\}_{n\in\Lambda}^{\infty}$

with $\int_M |\mathfrak{a}_n|^2$ bounded.

$\Rightarrow$ $\exists$ subs. $\Lambda' \subset \Lambda$ s.t. plus
$\{h_n'\} \in \mathrm{Aut}\,\mathbb{P}$ with
$\{h_n'^*\mathbb{A}_n\}$ conv. weakly in L^2_1
to L^2_1 connection on
$\mathbb{P}$.

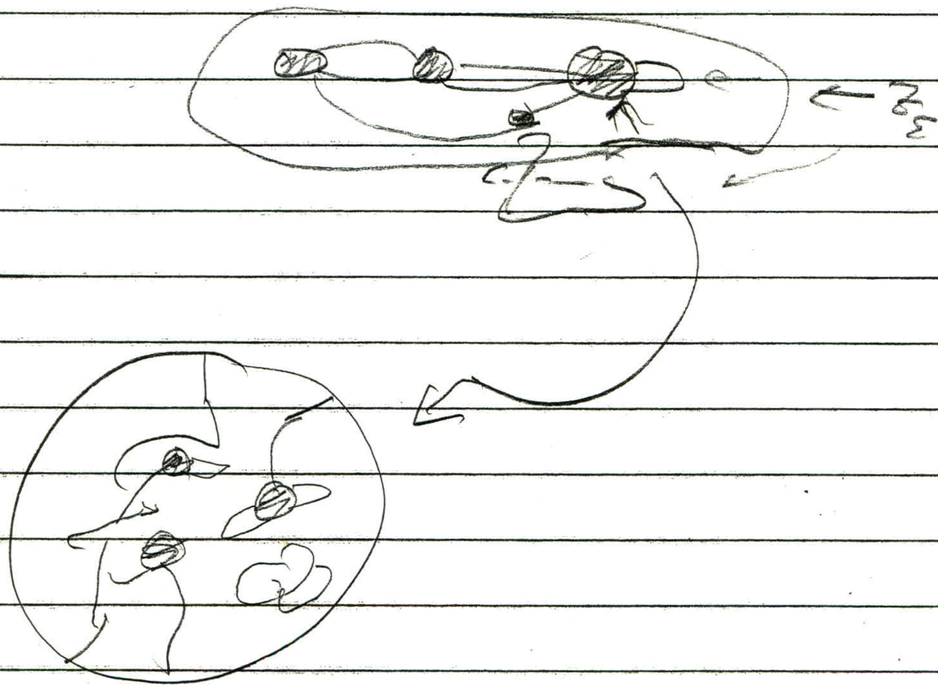
(2) $\exists$ no subsequence for (1).
In this case, $\exists$ L^2_1
1) $\frac{1}{2}$-Hölder continuous fnct $f: M \to \mathbb{R}$
2) $f^{-1}(0) = Z$, this is a closed, Hausdorff
dim ≤ 1 set contained
in a countable set of
Lipshitz curves
3) real line bundle $\mathcal{I} \to M \setminus Z$
3) harmonic $\mathcal{I}$-valued 1-form
$v \to M \setminus Z$ s.t. $|v| = f$,
$|\nabla v|^2 \in L^2$.

4) Picture of Z: Given $\varepsilon > 0$

$\exists$ $\mathcal{B}_\varepsilon$ = finite set of disjoint radius $\leq \varepsilon$ balls

Z_ε = finite length, embedded Lipschitz curve on $M - \bigcup_{B} \mathcal{B}$

$$Z = Z_\varepsilon \cup \left(\bigcup_{B \in \mathcal{B}} Z \cap B \right)$$



5) $\hat{A}$: Loc. connection $\hat{A}$ on $\mathbb{E}$

6) $\hat{A}$ - cov. constant L_2 section, σ

on $M \setminus Z$ of $(P \times_{\sigma} \mathbb{Z}) \otimes I$

7) Subs $A \subset \{1,2,\ldots\}$, $\{h_n\}_{n \in A} \subset Aut(P|_{M-Z})$

$h_n^* A_n = A_n + \nu \sigma_n$

a) $\{A_n\}$ conv weakly in $L^2_{1,loc}$ on $M-Z$ to $\hat{A}$

b) $\left\{\frac{\sigma_n}{|\sigma_n|}\right\}$ conv strongly in L^2_1-loc over $M-Z$

c) to $\nu \otimes \sigma$.

Special case: $M = \Sigma \times S^1$

all data input is S^1-invariant

$\Rightarrow$ ~~2n~~-d version

$Z = \Theta \times S^1$ $\quad$ Θ = $2g-4$ or fewer pts

genus Σ

ν^2 = hol. quadratic differential μ

ie: pts $\nu = \sqrt{\mu}$

4 = 2-fold branched cover given by $\sqrt{\mu}$.

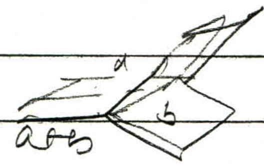
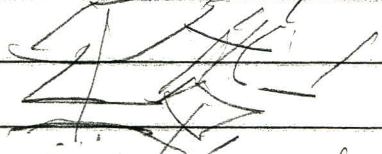
(trivial if & only if μ vanishes to even order at each 0

$\mu \sim z^p$ $\qquad$ $\nu \sim z^{p/2}$

1. Special case: $\{M_n\}$ are flat $PSL(2;\mathbb{C})$ connections.

- Morgan–Shalen comp. of $PSL(2;\mathbb{C})$ character variety $= Hom(\pi_1; PSL(2,\mathbb{C}))/PSL(2;\mathbb{C})$
  π_1-equiv. $\tilde{M} \to Y =$ $\mathbb{R}$-tree
  in term of maps to $\mathbb{R}$-trees:
  A ~~metric~~ space w/ prop that $\exists$ unique path between any 2-pts.

- Circle of ~~ideas~~ developed by MS, Gabai, Oertel, Hatcher, among many others.

  i) Branched surfaces 
  transverse-wtd (incomp
  ii) Singular, transv. measured foliations
  
  iii) transv. measured laminations
  iv) Maps (π_1-equiv) $u: \tilde{M} \to Y = \mathbb{R}$-tree

- Korevaar–Schoen: Morgan–Shalen map realized by harmonic, π_1-equiv map $u: \tilde{M} \to \mathbb{R}$.

a) X_n-Sun Korevaar–Schoen map has H-$d = 1$ singular set,

$Q \subset \tilde{M} = u|_{\tilde{M} \setminus Q} =$ ~~smooth~~ $du \neq 0$.

Q is closed, H-dim ≤ 1.

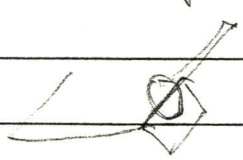
b) $\pi^{-1} Z \subset \pi^{-1} Q$

$\pi^* v|_{M - \pi^{-1}Q} = \pi^* du$

Top sign in general case.

- $\ker v \subset T(M \setminus Z)$ is integrable, Z-plane field, so tangent planes to transv. measured singular foliation.

- Can construct from leaves $+ v$
  i) a wtd branched surface in M much like that used by M-S (but not incompr)
  ii) Map from $\tilde{M}$ to $\mathbb{R}$-tree also.



$v = \mathrm{Re}\, d(z^{3/2}) = d\mu$, $\mu = \mathrm{Re}\, z^{3/2}$

near Z_Σ

$\mu^{-1}(0) =$ branched surface

⇔ Why timely?

1) Techniques used almost surely needed for

a) $PSL(2;\mathbb{C})$ Floer hom.

⇒ in general

$$\tau(F_A - \sigma\wedge\sigma)^+$$

$$\dot{A} = \alpha * (F - \sigma\wedge\sigma) + \beta * d_A\sigma = 0$$

$$\dot{\sigma} = \alpha * d_A \sigma + \beta * (F_A - \sigma\wedge\sigma) = 0$$

$$\alpha^2 + \beta^2 = 1$$

introduced by W. in context of geom. Langlands, interpr. of Khovanov homology.

($\alpha = 1$ case is Floer hom)

2) 4-mfld version

$$\tau(F_A - \sigma\wedge\sigma)^+ + (1-\tau)(d_A\sigma)^+ = 0$$

$$(1-\tau)(F_A - \sigma\wedge\sigma)^- + \tau(d_A\sigma)^- = 0$$

3.) Generalized SW-like mod. in $d = 3, 4$. eqn. $\left(\begin{matrix} *F_A = \psi^\dagger\tau\psi \\ D_A\psi = 0 \end{matrix} \right)$

$$*F_A = \psi_1^\dagger \tau \psi_1 + \psi_2^\dagger \tau \psi_2$$

$$D_A\psi_1 = D_A\psi_2 = 0$$

4) Vafa Witten eqn. in $d = 4$