Extensions of Karen Uhlenbeck's Theorem.

• S. Donaldson in 1980's

non-Abelian gauge theory

$$\begin{array}{c} G \subset P \\ \downarrow \\ M \end{array} \qquad \text{dim } 3,4.$$

G = Lie group
P = principal G-bundle.

$\{A_k\}_{k=1}^{\infty}$ connections on P

$SU(2)$ = 2×2 unitary matrices

$$A \Rightarrow F_A = \underline{\underline{dA}} + A \wedge A$$

A: matrix valued 1-form

F_A: matrix valued 2-form

$G = SU(2)$

<u>Thm:</u> $\{A_k\}_k$ with

$$\int_M |F_{A_k}|^2 < E < \infty$$

If $d=3$, $\exists \{g_k\}_{k=1}^{\infty}$ automorphism of P

such that: $\{g_k^* A_k\}_{k=1}^{\infty}$ has subsequence that conv. weakly in Sobolev L^2_1 - topology

$$g^*A = gAg^{-1} + g\,d(g^{-1})$$

If $d=4$: Same conclusion but conv. on compact sets in complement of finite set of pts

max # pts is determined by E

$$F_{g^*A} = g F_A g^{-1}$$

$$|F_A|^2 = -\,\mathrm{trace}\left((F_A)_{ij}(F_A)_{lm}\right) m^{il} m^{jm}$$

$$\int |F_A|^2 = \int |F_{g^*A}|^2$$

compact groups have a conjugation invariant norm.

Timely to consider
non-compact
groups

$SL(2;\mathbb{C})$

Analog of Uhlenbeck's theorem
$G = SL(2;\mathbb{C})$ for dim $= 3$.

$su(2) =$ Lie algebra of $SU(2)$
$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$sl(2;\mathbb{C}) = b + ia$ ← Lie alg $su(2)$

↑ Lie alg. of $su(2)$

$\mathbb{A} = \underset{\uparrow\ su(2)}{A} + i\underbrace{\mathfrak{a}}$ ← Hermitian part.

$F_{\mathbb{A}} = F_A - \mathfrak{a}\wedge\mathfrak{a} + i\, d_A\mathfrak{a}$

$d_A\mathfrak{a} = d\mathfrak{a} + [A, \mathfrak{a}]$

$$\boxed{|F_{\mathbb{A}}|^2 = |F_A - \mathfrak{a}\wedge\mathfrak{a}|^2 + |d_A\mathfrak{a}|^2}$$

$$\mathbb{A} = A + i\alpha$$

$$\mathcal{E}(\mathbb{A}) = \inf_{\mathbb{A}' = A' + i a'} \left(\int \left(|F_{A'} - \alpha' \wedge \alpha'|^2 + |d_{A'}\alpha'|^2 + |d_{A'} * \alpha'|^2 \right) \right)$$

$$\downarrow$$

$$\mathbb{A}' = g^* \mathbb{A}$$

g an Automorphism of principal $SL(2;\mathbb{C})$ bundle.

$$\alpha = \alpha_i \, dx^i$$

$$d_A * \alpha = \left(\partial_i a_i + [A_i, a_i] \right) dx^1 dx^2 dx^3$$

<u>Thm</u>: $\{\mathbb{A}_k\}_{k=1}^{\infty}$ with $\mathcal{E}(\mathbb{A}_k) < E < \infty$

$\mathbb{A}_k = A_k + i\alpha_k$

1) $\left\{ \int_M |\alpha_k|^2 < R^2 < \infty \right\}_{k=1}^{\infty}$

$h_k^* \mathbb{A}_k = h_k A_k h_k^{-1} + h_k d h_k^{-1}$

then there is a subsequence & corresponding sequence of automorphisms s.t. $\{h_k^* \mathbb{A}_k^*\}_{k \in \text{subs.}}$ conv. wkly in L^2_1-top.

$\{h_k\}$ = Aut of $SL(2;\mathbb{C})$ bundle

2) Suppose $\left\{ \int_M |\alpha_n|^2 \right\}_{n=1}^{\infty}$ diverges

$$r_n = \sqrt{\int_M |\alpha_n|^2}$$

$$A_n = A + i\alpha_n$$

$$a_n = \frac{1}{r_n}\alpha_n$$

~~There exists subsequence~~

~~+ a countable set of Lipschitz curves~~

(重來)

- $U \subset M$. Real line bundle $\mathcal{I} \to U$

- 1-form with values in $\mathcal{I}$

  ν

- Lipshitz curve in M.

  $\gamma : \mathbb{R} \to M$ such that

$$\begin{tikzcd} \mathbb{R} \arrow[r] & M \\ I \arrow[u, phantom, sloped, "\subset"] \arrow[ur] & \end{tikzcd}$$

$$\underbrace{|\gamma(t) - \gamma(t')|}_{\mathrm{dist}(\gamma(t), \gamma(t'))} < \overset{C}{\cancel{\text{const}}}\, |t - t'|$$

- $C^{0,1/2}$ — Holder norm

$$\sup_{x, y \in U} \frac{|f(x) - f(y)|}{\mathrm{dist}(x, y)^{1/2}}$$

1) $\mathbb{A}_n = A_n + \alpha_n$

$$a_n = \frac{1}{r_n}\alpha_n \qquad r_n = \left(\int_M |\alpha_n|^2\right)^{1/2}$$

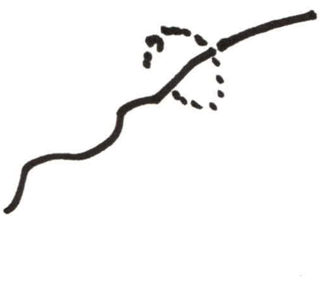
a) $C^{0,1/2}$ Holder continuous fuction
& L^2_1 $\quad f : M \to \mathbb{R}$

b) $f^{-1}(0) = Z$ is a closed set of
Hausdorff dim ≤ 1
contained in a countable
set of Lipshitz curves

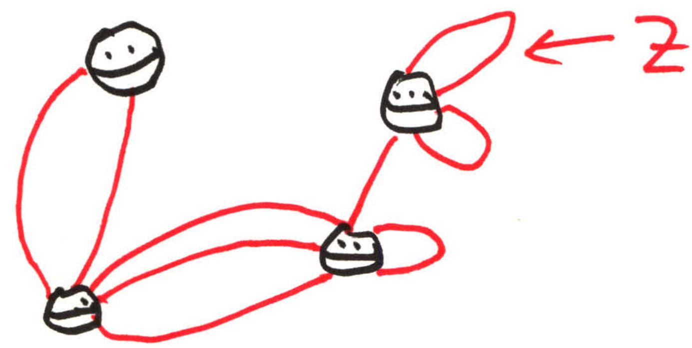
c) real line bundle $\mathcal{I} \to M \setminus Z$

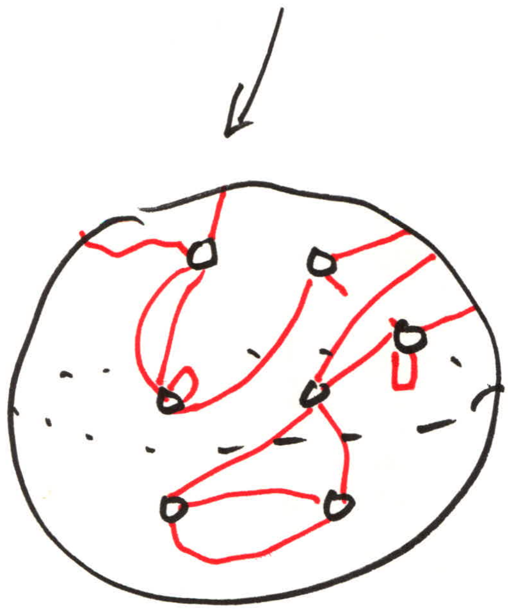
d) harmonic, $\mathcal{I}$-valued 1-form,
v on $M \setminus Z$ $\quad |v| = f$
$|\nabla v|^2 \in L^2(M)$

$dv = 0$
$d{*}v = 0$



Given $\varepsilon > 0$





- There exist $L^2_{1;loc}$ connection $\hat{A}$ on $M \setminus Z$ SU(2) connection
- $\hat{A}$-covariantly constant, norm 1 su(2) valued, function on $M \setminus Z$ ($\mathfrak{g}$-valued) σ $\quad d\sigma + [\hat{A}, \sigma] = 0$
- Sequence of SU(2) aut. $\{g_n\}$ on $M \setminus Z$ s.t.

$\{g_n^* A_n\}$ conv in wk $L^2_{1;loc}$ top to $\hat{A}$ on $M \setminus Z$

$\{g_n^* a_n = \Omega_n^{-1} g_n^* \circ \Omega_n\}$ converges strongly ~~m~~ in $L^2_{1,loc}$ on $M \setminus Z$ to

$V \otimes \sigma$