1. Thm: $M$ = compact Riem. 3-manifold

Sequence $\{A_h : A_h + \mathfrak{a}_h\}_{h=1}^{\infty}$

$SL(2;\mathbb{C})$ connection ($A_h = SU(2)$ conn, $\mathfrak{a}_h$ = Lie alg 1-form)

such that

$$\int_{M^3} \left( |F_{A_h} - \mathfrak{a}_h \wedge \mathfrak{a}_h|^2 + |d_{A_h} \mathfrak{a}_h|^2 + |d_{A_h} * \mathfrak{a}_h|^2 \right) < R^2$$

1) If $\{\int |\mathfrak{a}_h|^2\}_{h=1}^{\infty}$ has bounded subseq.

$\exists$ subseq. $\{h_k\}_{k=1}^{\infty}$ of $SU(2)$ automorphisms s.t.

$\{h_k^* A_k\}$ conv in $L^2_1$ wk topology

$\{h_k^* a_k\}$

2) If $\{\int |\mathfrak{a}_h|^2\}$ has no bounded subseq., $\exists$

a) $C^{0,1/2}$ & $L^2_1$ fncl. $f: M \to \mathbb{R}^3$

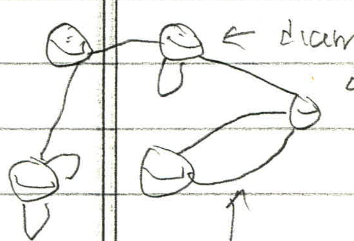
b) $f^{-1}(0) = Z$ is countable union of Lip. curves & Hausd. dim $\leq 1$

c) real line bundle $\mathcal{I} \to M \setminus Z$

d) $\mathcal{I}$-valued, harmonic 1-form $v$ on $M \setminus Z$ with $|v| = f$ & $|\nabla v| \in L^2_1(M)$.

e) $L^2_{1;loc}$ $SU(2)$ connection $\hat{A}$ on $M \setminus Z$

Structure of $Z$:




$Z$ = finite set of finite length Lip. curves on compl of balls.

f) $\hat{A}$ - cov. constant, $\mathfrak{g}$-valued map
$\sigma : M \setminus Z$ to Lie alg $su(2)$

g) subsequence of $\{(A_n, \alpha_n)\}$
and $SO(2)$ autom. $\{h_n : M \setminus Z \to SU(2)\}$
s.t.

1) $\{h_n^* A_n\}$ converges in $L^2_1$; weak, local top. to $\hat{A}$ on $M \setminus Z$

2) $\{\frac{1}{\Omega_n} \alpha_n\}$ converges in $L^2_{1;loc}$ on $M \setminus Z$ to $\nu \sigma$

$$\Omega_n^2 = \int_M |\alpha_n|^2$$

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2. Example: $M = S^1 \times \Sigma$, $\Sigma$ = Riem. surface

gives $d = 2$ version of Thm
by taking $\{(A_n, \alpha_n)\}$ to be $S^1$-inv.

- $\nu$ = $\mathfrak{g}$ valued 1-form on $\Sigma$

- write $T\Sigma \otimes \mathbb{C}$ as $T^{1,0} \oplus T^{0,1}$
$\nu$ as $\nu_{1,0} \oplus \nu_{0,1}$ $\quad \nu_{1,0} \in T^{1,0} \otimes \mathfrak{g}$

$\bar{\partial} \nu_{1,0} = 0$ $\quad \nu_{1,0}^2 = \mathrm{Hol}(T^{2,0})$ form

- $\nu_{1,0}^2$ has $4g-4$ zero's counting mult.
  $Z = S^1 \times$ zeros

- $\dim \mathrm{hol}(T^{2,0}) = 3g-3$

- Write $\nu_{1,0}^2 = \mu = \mathrm{hol}\ T^{2,0}$ section
  near zero of $\mu$, $\mu \sim z^p$

  $\nu_{1,0} \sim z^{p/2}$ so $\mathcal{I}$ is non-trivial near this zero if $p$ is odd, e.g $p=1$ is a simple zero.

3. Example $d=3$ $\{\mathbb{A}_h = A_h + i\alpha_h\}$ are all flat $SL(2;\mathbb{C})$ connections

$F_A = \alpha \wedge \alpha$
$d_A \alpha = 0$

- $(A + i\alpha)$ defines homomorphism $\pi_1(M) \to SL(2;\mathbb{C})$

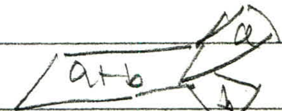
- Morgan–Shalen compactification of variety $\mathrm{Hom}(\pi_1, SL(2;\mathbb{C}))/\text{conj}$.

- boundary given by $\pi_1$-equiv. maps from $\hat{M}$ = univ. cover to $\mathbb{R}$-tree (contractible space w/ unique arc between any 2-pts)

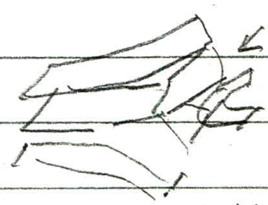
  example: • simplicial tree
  • comb

- Circle of ideas due to Morgan–Shalen, Gabai, Oertel, Hatcher, others

1) branched surface (incompressible) with weights




2) Singular, transversally measured foliations




3) Transversally measured laminations
4) $\pi_1$-equiv. maps to $\mathbb{R}$-trees of sort given by M–S

- Korevaar–Schoen: MS map realized by $\pi_1$-equiv. harmonic map to $\mathbb{R}$-tree.

  $u : \hat{M} \to Y = \mathbb{R}$-tree.

- X.–Sun Korevaar–Schoen map has closed, Hausd. dim $\leq 1$ singular set $Q \subset \hat{M}$ $\quad u|_{\hat{M} \setminus Q}$ has $du \neq 0$

- $\pi^{-1} Z \subset \hat{M}$ contained in $Q$

  $\pi^* \nu$ on $\hat{M} \setminus Q$ is $du$.

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Top. significance of $Z$ in general case.

- $\ker \nu \subset TM \setminus Z$ is integrable 2-plane field, so

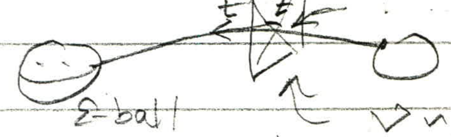
  tangent planes to transv. measured foliation.

- Can construct from leaves of $\nu$

  i) a wtd branched surface (leaves that limit in finite length curves to $Z$) a.c. Sun

  ii) Map from $\hat{M}$ to $\mathbb{R}$-tree equivariant (but not nec. of MS-type)

  iii)




$\nu \sim z^{1/p}$ $\quad u \sim z^{\frac{p+1}{p}}$

$= du$

foliation = Leaf = inverse image real(u).

Potential applications:

1.) $SL(2;\mathbb{C})$ - Floer homology.

- Chain complex generated by critical pts of $\int_M |\Omega|^2$ on space of

flat $SL(2;\mathbb{C})$ conn up to $SU(2)$ autom.

$$F_A - \Omega\wedge\Omega = 0$$
$$d_A\Omega = 0$$
$$d_A{*}\Omega = 0$$

- Differential "counts" solutions:
$\mathbb{A} = A + i\Omega$ on $\mathbb{R}\times M$

$$\frac{\partial A}{\partial t} \pm *(F_A - \Omega\wedge\Omega) = 0$$
$$\frac{\partial\Omega}{\partial t} - *d_A\Omega = 0$$

s.t. $\lim_{|t|\to\infty} \mathbb{A}|_t$ goes to flat connection w/ $d{*}\Omega = 0$.

- Must prove set of such solutions is compact modulo $SU(2)$ Aut. to obtain well defined count

2.) Other 3+4d equations where very similar techniques should be necessary to define Donaldson like invariants by "counting" equiv. classes of solutions in alg. way.

1). dim = 4   Witten proposed family of equ.

i) $$\frac{\partial A}{\partial t} + \alpha * (F - \Omega \wedge \Omega) - \beta * d_A \Omega = 0$$

$$\frac{\partial \Omega}{\partial t} = \alpha * d_A \Omega - \beta * (F_A - \Omega \wedge \Omega) = 0$$

$$\alpha^2 + \beta^2 = 1$$

a) Introduced in W's description of geometric Langlands correspondence

b) Used by W with $[0,\infty) \times M$ to describe Khovanov homology of knots & links

ii) $SL(2;\mathbb{C})$ like eq. on Compact 4-manifolds

a) $P_+(F_A - \Omega \wedge \Omega) = 0, \quad P_- d_A \Omega = 0$

$d_A * \Omega = 0$

b.) $\alpha, \beta$ version.

$$P_+(\alpha(F_A - \Omega \wedge \Omega) - \beta d_A \Omega) = 0$$

$$P_-(\alpha(d_A \Omega) + \beta(F_A - \Omega \wedge \Omega)) = 0$$

$$d_A * \Omega = 0$$

c) Vafa-Witten equ. for $SU(2)$-connection + Lie alg section of $\Lambda^+$:

$$P_+(F_A - \omega \# \omega) = 0 \qquad d_A \omega = 0$$

iii) Generalization of 3-d thm from $SL(2;\mathbb{C})$ to $SL(n;\mathbb{C})$

Complication in fact that dim maximal torus $> 1$

iv) Generalized Seiberg-Witten eq.

- original had $*F_A = \psi^+ c\, \psi$
  $D_A \psi = 0$
  $A = U(1)$ connection
  $\psi =$ spinor w/ values in assoc. line bundle
  $D_A =$ Dirac op.
  $\psi c \psi =$ adjoint of Clifford mult. map.

- Generalize to case with $> 1$ # spinors
  - $*F_A = \psi_1^+ c\, \psi_1 + 2\psi_2^+ c\, \psi_2$
    $D_A \psi_1 = D_A \psi_2 = 0$
  - More than 1 connection also
- 4-d analogs.

Input to proof.

1.) Local version of Uhlenbeck's thm: $d = 3$ $B \subset M$ a ball $\{A_n\}$ = connections on $B$

$$\int_B |F_{A_n}|^2 \leq E < \infty$$

$\Rightarrow$ subseq. with automorphisms $\{h_n\}$ s.t. $h_n^* A_n$ converges weakly in $L^2_1$-top. on $B_1$

2) Bochner-Weitzenbock formula for

$$\int_M |d_A \alpha|^2 + |d_A{}^* \alpha|^2$$

$$\shortparallel$$

$$\int_M \Big( |\nabla_A \alpha|^2 + Ric(\alpha, \alpha) + \langle F_A \wedge x(\alpha \wedge \alpha) \rangle \Big)$$

3) Bochner-W + Uhlenbeck give thm when $\{\alpha_n\}$ is bounded in $L^2$.

i) Bochner: $\int_M |\nabla_{A_n} \alpha_n|^2$ bounded

$\Rightarrow \int_M |d|\alpha_n||^2$ bounded

$\Rightarrow$ $|\alpha_n|$ bounded in $L^6$, $L^4$

$\Rightarrow$ $\sigma_n \wedge \sigma_n$ bounded in $L^2$

$\Rightarrow$ $F_{A_n}$ bounded in $L^2$

$\Rightarrow$ apply Uhlenbeck to get automorphisms $\{h_n\}$.

4) Bochner-Weitz.

$$Q_{A_t}(\sigma_t) = *d_A * d_A \sigma + d_A * d_A * \sigma = -\nabla_A^2 \sigma + Ric(\sigma, \sigma) + *[*F_A, \sigma]$$

5) Heat eq. $\partial_t \hat{\sigma}_n = -Q_{A_n}(\hat{a}_n)$
to get new seq.
$\{(A_n, \hat{a}_n)\}$ with
$\|Q_{A_n}(\hat{\sigma}_n)\|_2 \le \varepsilon_0 \|\sigma_n\|_2$

6) Frequency function of Almgren
(Han-Hardt-Lin

$$N(r) = \frac{r \int_{B_r} (|\nabla_A \sigma|^2 + |\sigma \wedge \sigma|^2)}{\int_{\partial B_r} |\sigma|^2}$$

- Frequency function used to prove reg. of harmonic fnc, control singular set of zero locus of eigenvectors of Laplacian & related eq.

e.g. $N = \frac{r\int |\nabla f|^2}{\int_{\partial B_r} |f|^2}$

Key point: $N$ is nearly non-increasing

1) $\frac{dN}{dr} \geq 0$.

2) $N(r)$ for $r \to 0$ is $p$ if $f \sim r^p$ near $r=0$.

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Frequency function used in 2 ways

1) to prove $Z$ = closed set using non-Abelian A version
2) to prove properties of $Z$ using version for harmonic $\mathbb{Z}$-valued form $\nu$.