· Potential Applications:

1.) $SL(2;\mathbb{C})$ Floer homology

- Chain complex: generated over $\mathbb{Z}$ by critical pts of $\int |\alpha_{4}|^2$ on $SL(2;\mathbb{C})$ flat connection moduli space $\left\{ A : \begin{array}{l} F_A - \alpha\wedge\alpha = 0, \\ d_A \alpha = 0 \\ d_A * \alpha = 0 \end{array} \right\}$
- Differential: Counts solutions on $\mathbb{R}\times M$ $(d=4)$ to
  $$\frac{\partial A}{\partial t} + *(F_A - \alpha\wedge\alpha) = 0$$
  $$\frac{\partial \alpha}{\partial t} \mp * d_A \alpha = 0$$
  (note: $d_A * \alpha$ pres. by flow)

- Solutions must limit as $t \to \pm\infty$ to flat $SL(2;\mathbb{C})$ connection
- To have well defined count, must have finite # solutions limiting $t \to -\infty$ to a given flat connection.
- Prove this: Assume $\{A_n\}_{n=1}^{\infty}$ is sequence of sol. and analyze limit.

- Why $SL(2;\mathbb{C})$? $SL(2;\mathbb{C})$ flat connections seem to play a larger role in classical 3-d top. than do $SU(2)$. Perhaps Floer hom. gives more info. also.

2) - Related equations 3 & 4 dim equations proposed by Witten to give Khovanov homology of knots & links.

on $[0,\infty)\times M$ or $\mathbb{R}\times M$

$$\boxed{\begin{aligned} &\dot{A} + \alpha * (F - \Omega\wedge\Omega) - \beta * d_A \Omega = 0 \\ &\dot{\Omega} - \alpha * d_A \Omega - \beta * (F - \Omega\wedge\Omega) = 0 \\ &\qquad\qquad\qquad \alpha^2+\beta^2 = 1 \\ &d_A * \Omega = 0 \end{aligned}}$$

limit $t \to \pm\infty$ Flat

$\Rightarrow$ related to symplectic/hyper Kähler structure on space of $SL(2;\mathbb{C})$ connections.

- Note: $d_A * \Omega$ preserved by flow.

- These are gradient flow equations for "slice" of Chern-Simons

- $$CS = \int_M tr\left(\mathbb{A}\wedge F_{\mathbb{A}} - \tfrac{1}{3}\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A}\right)$$

$f = \mathrm{Re}\left((\alpha+i\beta)\, CS\right) \qquad (\dot{A},\dot{\Omega}) = -\nabla f.$

$$\omega = \int_M tr\Big(A\wedge F_A - \tfrac{1}{3} A\wedge A\wedge A - \phi\wedge d_A\phi - \phi\wedge\phi\wedge A\Big)$$

$$+\, i\int_M tr\Big(\phi\wedge F_A + A\wedge d_A\phi - \phi\wedge A\wedge A + \tfrac{1}{3}\phi\wedge\phi\wedge\phi\Big)$$

Related 4-d equations

1) Counting solutions algebraically may give interesting "Donaldson" invariants $\quad \Lambda^2 T^* = \Lambda^+ \oplus \Lambda^-$

- $(F_A - \phi\wedge\phi)^+ = 0 \qquad (d_A\phi)^- = 0$

  $d_A * \phi = 0$

- $\alpha(F_A - \phi\wedge\phi)^+ - \beta(d_A\phi)^+ = 0$

  $\alpha(d_A\phi)^- + \beta(F_A - \phi\wedge\phi)^- = 0$

  $d_A * \phi = 0$

- "Counting" requires finite # solutions

2) Equations (Vafa-Witten) for $SU(2)$ connection & $su(2)$-valued section of Λ^+

$$(F_A = W \# W)^+ = 0$$

$$d_A W = 0$$

3) Generalized Seiberg-Witten equations. (Very analogous analysis should be applicable)

- $*F_A = \Psi^{+}\tau\Psi$ dual to Cliff mult.

  $D_A\Psi = 0$

  ↑ Dirac equ. $\Psi = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$

$$\begin{pmatrix} i\nabla_3\alpha + i(\nabla_1 - i\nabla_2)\beta = 0 \\ -i\nabla_3\beta + i(\nabla_1 + i\nabla_2)\alpha = 0 \end{pmatrix}$$

$$(*F)_3 = i(|\alpha|^2 - |\beta|^2)$$

$$(*F)_1 = i(\alpha\bar{\beta} + \bar{\beta}\alpha)$$

$$(*F)_2 = (\alpha\bar{\beta} - \bar{\beta}\alpha) \qquad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

- Solution space is compact modulo $C^\infty(M; iu(1))$
- Case of 2-spinors:

$$D_A\Psi_1 = 0, \; D_A\Psi_2 = 0$$

$$F_A = \Psi_1^{+}c\Psi_1 + \Psi_2^{+}c\Psi_2$$

Sol. space compact if

$$\int(|\Psi_1|^2 + |\Psi_2|^2) < R < \infty.$$

- Case of more spinors &/or more connections.
- 4d Analogs to give analogs of SW-invariants

Input to the proof

1) Uhlenbeck's theorem: (local version)

$U \subset M$ = open set

$$\left\{ \int_U |F_{A_n}|^2 \right\}_{n=1}^{\infty} < E < \infty$$

$\Rightarrow$ $\exists$ subsequence plus $\{ h_n : U \to SU(2) \}$ s.t.

$$\{ h_n A_n h_n^{-1} + h_n d h_n^{-1} \}_{n=1}^{\infty}$$

converges weakly in $L^2_{1,loc}(U)$

2) Weitzenbock formula:

$$\int_M \left(|d_A \alpha|^2 + |d_A * \alpha|^2 \right) =$$

a) $$\int_M \left(|\nabla_A \alpha|^2 + 2 \langle F_A \wedge * \alpha \wedge \alpha \rangle + Ric(\alpha, \alpha) \right)$$

- Gives proof of case 1. since

b) $$\int_M |F_A - \alpha \wedge \alpha|^2 = \int_M \left(|F_A|^2 - 2 \langle F_A \wedge \alpha \wedge \alpha \rangle + |\alpha \wedge \alpha|^2 \right)$$

a)+b): $$\int_M |\nabla_A \alpha|^2 + |F_A|^2 + |\alpha \wedge \alpha|^2 + Ric(\alpha, \alpha) < 2E < \infty$$

Thus if $\int |\sigma|^2 < R^2 < \infty$

$\Rightarrow \int |F_A|^2 < |Ric|_\infty R^2 + E$

$\Rightarrow$ A converges in L^2_1 wh.

$\int |\nabla_A \sigma|^2 < |Ric|_\infty R^2 + E$

$\Rightarrow$ σ conv. in L^2_1 wh.

ii) In case 2: $\{\int |\sigma_n|^2\}$ diverges, leads formally to conclusion that

$a_n = \frac{1}{\sqrt{2}_n} \sigma_n$ converges to $\nu \otimes \sigma$

$$\int_M \sqrt{2}_n^2 |a_n \wedge a_n|^2 \leq C < \infty$$

$$\Rightarrow \int_M |a_n \wedge a_n|^2 \leq \frac{C}{\sqrt{2}_n^2} \to 0$$

$$\left(\int_M (|d_{A_n} a_n|^2 + |d_{A_n} * a_n|^2) \leq \frac{C}{\sqrt{2}_n^2} \to 0 \right.$$

$a_n \rightsquigarrow \hat{a}$ $\quad \hat{a} \wedge \hat{a} = 0$

$\Rightarrow \hat{a} = \nu \sigma$ $\quad |\sigma| = 1$

$$\left. \begin{array}{l} d_A \hat{a} = 0 \\ d_A * \hat{a} = 0 \end{array} \right\} \Rightarrow \begin{array}{l} d\nu = 0 \\ d * \nu = 0 \end{array} \quad d_A \sigma = 0.$$

3.) In Case 2: $\left\{\int |\alpha_n|^2\right\} \to \infty$

Weitzenbock formula:

- $\Rightarrow$ (1) $\int |\nabla_{A_n} a_n|^2 + r^2 |a_n \wedge a_n| \leq C_0$

  [2) $\frac{1}{r^2}\int |F_{A_n} - r^2 a_n \wedge a_n|^2 \leq C_0$

  $\Rightarrow$ $a_n \to$ in L^p

  $|a_n| \to$ in L^2_1 weakly

  $a_n \wedge a_n \to 0$ in L^2

- Can use heat eq to perturb α_n by $O(r^{-1})$ in L^2_1

  to $\hat{\alpha}_n$ with

  $\frac{1}{r_n}\hat{\alpha}_n = \hat{a}_n$ such that

  $\sup |\hat{a}_n| \leq C_0$

- $\Rightarrow$ $\lim \hat{\alpha}_n \in L^\infty$

- define ptwise by

  $f = |\hat{a}_\infty|(x) = \limsup_{k \to \infty} |\hat{\alpha}_n|(x)$

4) Almgren's frequency function:

used to study critical set of harmonic functions, nodes of eigenvalues of Laplacian

$$-\Delta f = \lambda f \qquad f^{-1}(0) \text{ is singular ?}$$

study

$$N_f(r) = \frac{r \int_{B_r} |\nabla f|^2}{\int_{\partial B_r} |f|^2}$$

$$\boxed{\frac{d}{dr}\left(\frac{1}{r^{d-1}} \int_{\partial B_r} f^2\right) = \frac{2N}{r} \left(\frac{1}{r^{d-1}} \int_{\partial B_r} f^2\right)}$$

⟹ Introduced by Almgren

⟹ exploited by others (Han-Hardt-Lin)

⟹ in Euclidean space, N is increasing funct. of r

⟹ in M, N is "essentially increasing"

$$\frac{dN}{dr} \geq -C_0 r^2.$$

example: $f(x) \sim x^p$ for $x \sim 0$

$N \sim p$ as $r \to 0$

$$\left(\int_{\partial B_{r_1}} f^2\right)(r_1) \sim \left(\frac{r_1}{r_0}\right)^p \left(\int_{\partial B_{r_0}} f^2\right)$$

4) Almgren's frequency function

$p \in M$, $r \in [0, C_0^{-1})$

<u>Non Abelian version</u>:

- $$N_p(r) = \frac{r \int_{B_r} (|\nabla_A a|^2 + r^2 |a \wedge a|^2)}{\int_{\partial B_r} |a|^2}$$

Thm: $N_p(r)$ is essentially increasing

used to prove $Z = f^{-1}(0)$ is closed set & f is continuous

- Version for ν

$$N_p(r) = \frac{r \int_{B_r} |\nabla \nu|^2}{\int_{\partial B_r} |\nu|^2}$$

~~also~~ essentially increasing

used to prove structure thm for $f^{-1}(0)$. (Hausd. dim, Lipschitz curves