Potential Application

- $SL(2;\mathbb{C})$ Floer homology

Chain complex: Critical points of function on $SL(2;\mathbb{C})$ flat connections (modulo action maps to $SU(2)$)

of $\mathbb{A} = A + i\alpha \to \int_M |\alpha|^2$

$$F_A - \alpha\wedge\alpha = 0$$

$$d_A\alpha = 0 \qquad A \to hAh^{-1} + hdh^{-1}$$

$$d_A{*}\alpha = 0 \qquad \alpha \to h\alpha h^{-1}$$

$$h : M \to SU(2)$$

- Differential

~~state~~ $\mathbb{A}_t \quad t \in \mathbb{R}$

$$\frac{\partial A}{\partial t} + *(F_A - \alpha\wedge\alpha) = 0$$

$$\frac{\partial \alpha}{\partial t} - * d_A\alpha = 0$$

$d_A{*}\alpha = 0$

$t \to \pm\infty$, solution goes to flat $\mathbb{A}$.

$\{\mathbb{A}_k\}_{k=1}^{\infty}$

· Witten: Geometric Langlands theory.

· : Khovanov homology

$\mathbb{A} = A + i\Omega \qquad \alpha^2 + \beta^2 = 1$

$$\frac{dA}{dt} + \alpha * (F_A - \Omega\wedge\Omega) - \beta * d_A\Omega = 0$$

$$\frac{d\Omega}{dt} - \alpha * d_A\Omega - \beta * (F_A - \Omega\wedge\Omega) = 0$$

$\mathbb{A}|_t \qquad \mathbb{R}\times M \qquad d_A * \Omega = 0$

$[0,\infty)\times M$

Gradient flow equations
Chern-Simons functional

$\mathbb{A} = A + i\Omega$

$$CS \quad \int_M tr(\mathbb{A}\wedge F_{\mathbb{A}} - \tfrac{1}{3}\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A})$$

$$real(CS) = \int_M tr(A\wedge F_A - \tfrac{1}{3}A\wedge A\wedge A - \Omega\wedge d_A\Omega - \Omega\wedge\Omega\wedge A)$$

$$im(CS) = \int_M tr(\Omega\wedge F_A + A\wedge d_A\Omega - \Omega\wedge A\wedge A + \tfrac{1}{3}\Omega\wedge\Omega\wedge\Omega)$$

$$f = \text{real}\left((\alpha + i\beta)\, CS\right)$$

$\mathbb{R} \times M$ $\qquad \{\mathbb{A}_t\}_{t \in \mathbb{R}}$

X = 4-dim, Riemannian + oriented.

$$\Lambda_2 T^* = \Lambda^+ \oplus \Lambda^-$$

Λ^+: $\omega = *\omega$ $\qquad$ Λ^-: $\omega = -*\omega$

$$\alpha (F_A - a \wedge a)^+ - \beta (d_A a)^+ = 0$$

$$\alpha (d_A a)^- + \beta (F_A - a \wedge a)^- = 0$$

$$d_A * a = 0$$

$$\mathbb{A}_n = \{(A_n, a_n)\}_{n=1}^{\infty}$$

modulo SU(2) automorphisms

Vafa–Witten

(A, ω) $A = SO(2)$ connection on X

ω is a Lie alg. valued self-dual 2-form

$$F_A^+ = (\omega \# \omega)^+$$
$$d_A \omega = 0$$

Λ^+ $\omega = \sum_{a=1}^{3} \omega_a e^a$

$$F_A^+ = F_{A,a}^+ e^a$$

$$F_{A,1}^+ = [\omega_2, \omega_3]$$
$$F_{A,2}^+ = [\omega_3, \omega_1]$$
$$F_{A,3}^+ = [\omega_1, \omega_2]$$

Generalized S-W equations

M^3 $\qquad$ $U(1)$-connection A

Ψ section of Spin_c bundle

determine

$\Psi = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ $\qquad$ α, β : Complex Functions

<u>$D_A\Psi = 0$</u> $\qquad$ $\begin{pmatrix} i\nabla_{A,z}\alpha + i(\nabla_{A,x}\beta - i\nabla_{A,y}\beta) \\ -i\nabla_{A,z}\beta + i(\nabla_{A,x}\alpha + i\nabla_{A,y}\beta) \end{pmatrix}$

x, y, z $\qquad$ $D_A\Psi$

$$*F_A = \Psi^{+}_{c}\Psi$$

$$(*F_A)_z = i(|\alpha|^2 - |\beta|^2)$$

$$(*F_A)_x = i(\alpha\bar\beta + \bar\alpha\beta)$$

$$(*F_A)_y = \alpha\bar\beta - \bar\alpha\beta$$

A, ψ_1, ψ_2

$$D_A \psi_1 = 0, \; D_A \psi_2 = 0$$

$$* F_A = \psi_1^{+} \tau \psi_1 + \psi_2^{+} \tau \psi_2$$

$SL(2; \mathbb{C})$

$SL(n; \mathbb{C})$

Input to Proof

1) Uhlenbeck's thm (local version)

$U \subset M \quad \left\{ \int_U |F_{A_n}|^2 < E < \infty \right\}_{n=1}^{\infty}$

$\exists$ subsequence & $\{ h_n : U \to SU(2) \}$

such that $\{ h_n A_n h_n^{-1} + h_n d h_n^{-1} \}$

converges on compact subsets of U in L_1^2-weak.

2) Weitzenbock formula:

a)
$$\int_M \left(|d_A \alpha|^2 + |d_A * \alpha|^2 \right) = \int_M \left(|\nabla_A \alpha|^2 + 2 \langle F_A \wedge *(\alpha \wedge \alpha) \rangle + \mathrm{Ric}(\alpha, \alpha) \right)$$

$\langle \cdot \rangle = -\mathrm{tr}(\cdot)$

b)
$$\int_M |F_A - \alpha\wedge\alpha|^2 = \int_M \left(|F_A|^2 + |\alpha\wedge\alpha|^2 - 2\langle F_A \wedge *(\alpha\wedge\alpha)\rangle\right)$$

c)
$$\int_M \left(|d_A\alpha|^2 + |d_A{*}\alpha|^2 + |F_A - \alpha\wedge\alpha|^2\right)$$
$$= \int_M \left(|\nabla_A\alpha|^2 + |F_A|^2 + |\alpha\wedge\alpha|^2 + Ric\langle\alpha,\alpha\rangle\right)$$

$$\int_M \left(|d_A \alpha|^2 + |d_A^* \alpha|^2 + |F_A - \alpha \wedge \alpha|^2 \right) \leq E^2$$

$$\int_M |\alpha|^2 \leq R^2$$

d) $$\Longrightarrow \int_M \left(|\nabla_A \alpha|^2 + |F_A|^2 + |\alpha \wedge \alpha|^2 \right) \leq E^2 + \sup |Ric| \, R^2$$

$$\int \left(|d_A \Omega|^2 + |d_A^* \Omega|^2 + |F_A - \Omega \wedge \Omega|^2 \right) \le E^2$$

Case 2) $\quad \Omega^2 = \int |\Omega|^2$

$$a = \frac{1}{\Omega} \Omega$$

$$\int \left(|d_A a|^2 + |d_A^* a|^2 + |F_A - \Omega^2 a \wedge a|^2 \right) \le \frac{1}{\Omega^2} E^2$$

$$\int_M \left(|\nabla_A a|^2 + \frac{1}{\Omega^2} |F_A|^2 + \Omega^2 |a \wedge a|^2 \le \frac{1}{\Omega^2} E^2 \right.$$

$+ \sup |Ric|$

$a_k \to \hat{a}$ in L^2_1

$$\hat{a} \wedge \hat{a} = 0$$

$$d_A \hat{a} = 0 \qquad d_A^* \hat{a} = 0$$

1-form

$\downarrow$

$\hat{a} = \nu \sigma$

$\uparrow$

Lie alg

$|\sigma| = 1$

$d\nu = 0 \quad d^*\nu = 0 \quad \nabla_A \sigma = 0$

Frequency function

Almgren

$$\boxed{\Delta f = 0}$$

$$\Delta f = \lambda f$$

$$f = 0 \qquad df = 0$$

Han-Hardt-Lin $\quad p \in M$

$[0, \varepsilon^{m})$

$$N(r) = \frac{r \int_{B(r)} |\nabla f|^2}{\int_{\partial B(r)} f^2}$$

$$\boxed{\frac{dN}{dr} \geq 0} \sim -cr^2$$

$|f| \sim |x|^p$

$N \sim p$ for $r \sim 0$

~~$K = \frac{1}{r^{d-1}} \int_{\partial B(r)} f^2$~~

$$K(r) = \frac{1}{r^{d-1}} \int_{\partial B_r} |f|^2$$

$$\frac{dK}{dr} = 2\frac{N}{r} K \qquad K(r) \geq \left(\frac{r}{r_0}\right)^{2p} K(r_0)$$

$r \geq r_0$

Non-Abellian Freq.

$$\Rightarrow \quad N(r) = \frac{r \int_{B(r)} \left(|\nabla_A a|^2 + 2\Omega^2 |a \wedge a|^2 \right)}{\int_{\partial B(r)} |a|^2}$$

~~$\int |F_A - \Omega \wedge \Omega|^2 + \int |d_A \Omega$~~

$$\frac{E^2}{\Omega^2} \geq \int \left(\frac{1}{\Omega^2} |F_A - \Omega^2 a \wedge a|^2 + |d_A a|^2 + |d_A * a|^2 \right)$$

$$r \geq \frac{c}{\Omega}$$

$$\Rightarrow \quad N(r) = \frac{r \int_{B(r)} |\nabla v|^2}{\int_{\partial B(r)} |v|^2}$$

$a \longrightarrow \hat{a}$ converges in L^∞

$$|\hat{a}| = \limsup_{n \to \infty} |\hat{a}_n|$$

$$\hat{a}_n = a_n$$

$$\frac{d}{dr}\left(\frac{1}{r^2}\int_{\partial B_r}|a|^2\right) = 2\frac{N}{r}\left(\frac{1}{r^2}\int_{\partial B_r}|a|^2\right)$$

$$C \ge \frac{1}{r^2}\int_{\partial B_r}|a|^2 \ge \left(\frac{r}{r_0}\right)^p\left(\frac{1}{r_0^2}\int_{\partial B_{r_0}}|a|^2\right)$$

$$\frac{1}{r_0^2}\int_{\partial B_{r_0}}|a|^2 \le \left(\frac{r_0}{r_1}\right)^p C$$