

Minimal submanifolds: old and new

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Chen-Jung Hsu Lecture 1, Academia Sinica, ROC

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Plan of Lecture

Part 1: Volume, mean curvature, and minimal submanifolds

Part 2: Variational theory and recent applications

Part 3: Minimal surfaces and eigenvalue problems

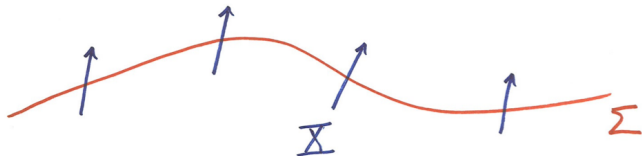
Part 1: Volume and mean curvature I

Given a curve in space we can measure its length, and given a surface we can measure its area. More generally we can measure the k -dimensional volume of a submanifold Σ^k in an n -dimensional ambient space M^n .

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Given a vector field X on M , we may choose a family of diffeomorphisms F_t whose derivative \dot{F} at $t = 0$ is equal to X . If we take $\Sigma_t = F_t(\Sigma)$, then how does the volume change?



Volume and mean curvature II

Assuming that X is orthogonal to Σ , we have

$$\frac{d}{dt}dV_t = -\langle \vec{H}, X \rangle dV \text{ at } t = 0$$

where \vec{H} is the *mean curvature* of Σ ; that is, for an orthonormal basis e_1, e_2, \dots, e_k tangent to Σ

$$\vec{H} = \sum_{i=1}^k \vec{S}(e_i, e_i)$$

where $\vec{S}(v, v)$ is the normal component of the curvature of a curve lying on Σ and tangent to v .

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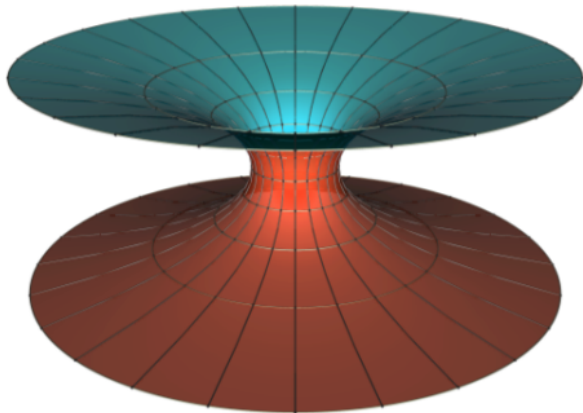
We see that the volume of Σ is stationary for all deformations if and only if $\vec{H} = 0$, and we call such a submanifold **minimal**. Note that minimal does not mean *volume minimizing*.

Examples I

Minimal surfaces have been studied for hundreds of years. Here are a few examples.

Examples I

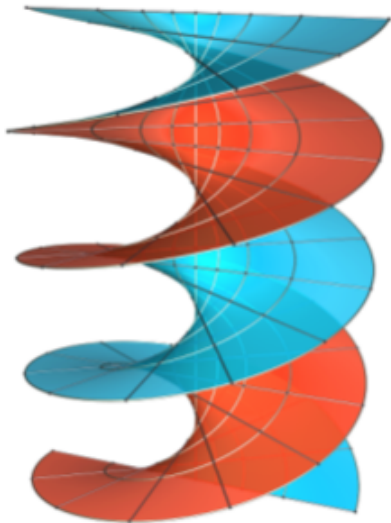
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This surface is called the catenoid. It was shown to be minimal by L. Euler in 1744.

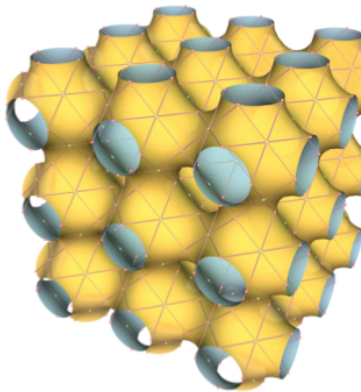
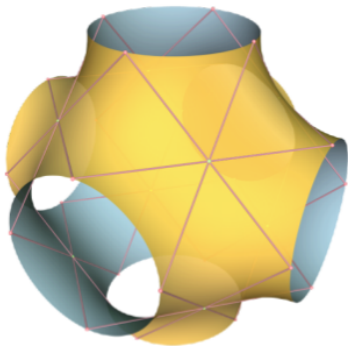
Examples II

This is the helicoid found by Euler in 1774.

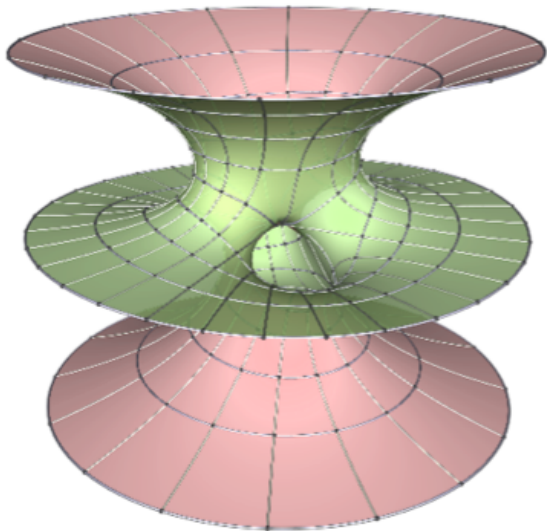


Examples III

This is a minimal surface discovered by H. A. Schwarz in the 19th century.

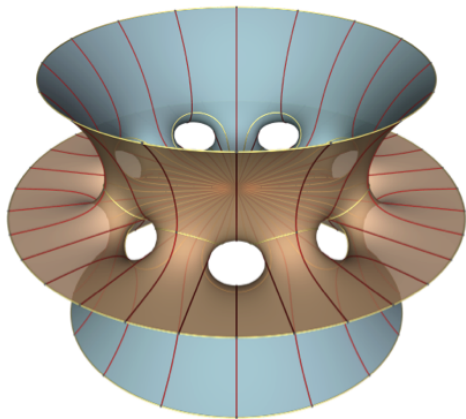


Examples IV



This surface was discovered by C. Costa and shown to be embedded by D. Hoffman and W. Meeks in the 1980s.

Examples V



These high genus Costa surfaces are a very special case of a general construction of N. Kapouleas from 1997.

A recent theorem on Minimal Surfaces in R^3

Notice that of the examples we have described, only the plane and the helicoid are simply connected. This means that any closed curve can be continuously shrunk to a point. This property is equivalent to being homeomorphic to the plane \mathbb{R}^2 .

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It was shown by Meeks and Rosenberg in 2005, refining work of Colding and Minicozzi, that the plane and the helicoid are the only simply connected properly embedded minimal surfaces in \mathbb{R}^3 .

Part 2: Variational theory and recent applications

The classical existence question for minimal surfaces is the **Plateau Problem** named after the Belgian physicist J. Plateau who lived in the early 19th century. He did extensive observations on the behavior of soap films and bubbles. If one neglects gravity, then soap films satisfy the variational principle that they minimize area for their boundary (at least among nearby surfaces).

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Plateau Problem: Given any reasonable simple closed curve Γ in \mathbb{R}^3 , find a surface Σ which bounds Γ and has least area among all surfaces bounding Γ .

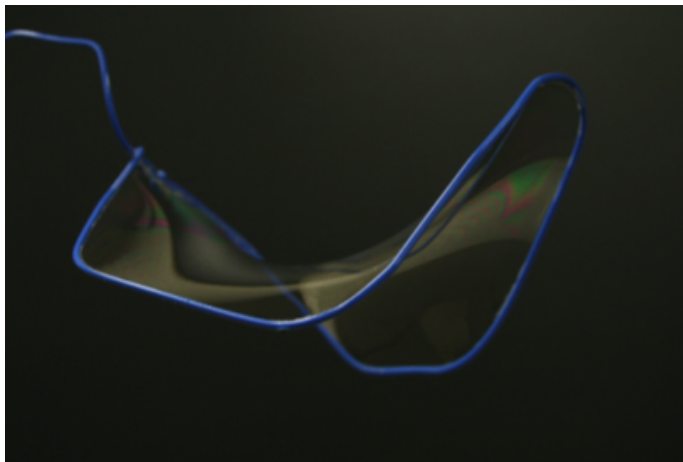
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To make this precise we have to specify the type of surface which we allow. For example, do we consider only orientable surfaces? Do we allow our surfaces to be singular? What does it mean to bound Γ for very general surfaces?

The Plateau problem



In the picture we see an area minimizer for the given boundary wire in \mathbb{R}^3 .

The generalized Plateau problem

The Plateau problem can be posed in any dimension and codimension and in an arbitrary curved space (Riemannian manifold).

Generalized Plateau problem: Given a closed $k - 1$ dimensional submanifold Γ which is the boundary of a k -dimensional submanifold, find a submanifold Σ^k of least volume among such bounding submanifolds.

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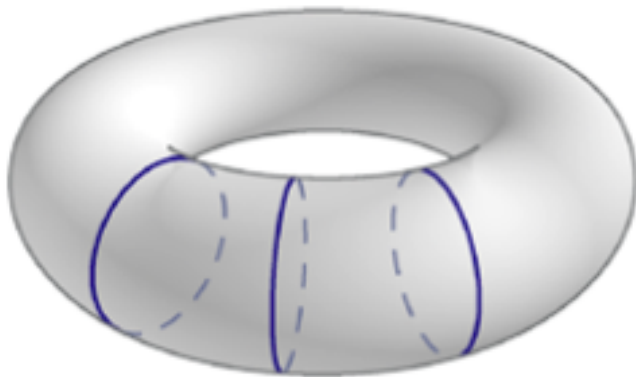
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A k -cycle is a closed oriented submanifold of dimension k . Two k -cycles Σ_1, Σ_2 are *homologous* if there is an oriented $k + 1$ -dimensional submanifold (possibly singular) B with $\partial B = \Sigma_1 - \Sigma_2$.

Minimizing in homology: Given a k -cycle Σ_0 , find k -cycle which is homologous to Σ_0 of least volume.

Volume Minimizing Cycles



In the picture we see minimizing curves in the homology class of curves which go one time around the handle.

Solvability of the problem

The Plateau problem for surfaces of disk-type in \mathbb{R}^n was solved by J. Douglas in 1930. He won the first Fields medal for the work in 1936. Although there have been many extensions of this work (surfaces with more general topologies, surfaces in Riemannian manifolds), the method is fundamentally two dimensional.

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A framework in which the generalized Plateau problem could be solved was finally developed with the theory of *integral currents* around 1960. This was done by H. Federer and W. Fleming. The minimizer is constructed as a limit of a minimizing sequence, and taking this limit requires completing the class of smooth oriented submanifolds in a topology for which the volume is continuous (or lower-semicontinuous). The price to be paid is that the solution is potentially very singular.

Regularity and singularities

The solution of the generalized Plateau problem was the impetus for the development of important methods to show the *partial regularity* of solutions of physical and geometric variational problems. There are many cases where singularities cannot be avoided such as for complex varieties of \mathbb{C}^n which are special solutions of the generalized Plateau problem.

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It was assumed that singularities for volume minimizing submanifolds could only occur when the codimension is greater than one. A great surprise in the subject, discovered through the work of several people by the late 1960s, was that for $k = n - 1$ and $n \geq 8$ singularities do sometimes occur.

Are Singularities Stable under Perturbations?

A fundamental question about minimizing hypersurfaces is the question of whether singularities are stable under perturbation of the data (either the boundary or the metric on M), or whether they can be perturbed away by a small deformation of the data. For example, suppose we fix a homology class in a smooth manifold M . We can then ask whether minimizers in this class can be singular for an open set of metrics on M .

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When $n = 8$ the singularities are isolated and in this case it was shown by N. Smale that they can be perturbed away. In higher dimensions we do not know the answer, but understanding a deeper reason for the existence of singularities would be very important.

Second variation

Just as in calculus, the question of whether a critical point is a local minimum can often be understood by looking at the second derivatives. We can compute the second variation of the volume when we deform a minimal submanifold along a vector field X . When $k = n - 1$ and there is an everywhere defined unit normal vector field ν along Σ , we can write $X = \varphi\nu$. The second derivative of the volume at $t = 0$ is given by the quadratic expression

$$\delta^2\Sigma(\varphi, \varphi) = \int_{\Sigma} [\|\nabla\varphi\|^2 - (\|A\|^2 + Ric(\nu, \nu))\varphi^2] dv$$

where φ is of compact support. Here $\|A\|^2$ is the square length of the second fundamental form (sum of squares of principal curvatures), and Ric is the Ricci curvature of the ambient manifold (will be explained in tomorrow's lecture).

Stability and Morse index

The second variation is related to an important differential operator called the Jacobi operator

$$L\varphi = \Delta\varphi + (\|A\|^2 + Ric(\nu, \nu))\varphi.$$

In fact we have

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If Σ is compact with or without boundary, the number of negative eigenvalues of L is finite and is called the **Morse index** of Σ . If each compact subdomain of Σ has Morse index zero, then we say that Σ is **stable**.

Geometry and second variation

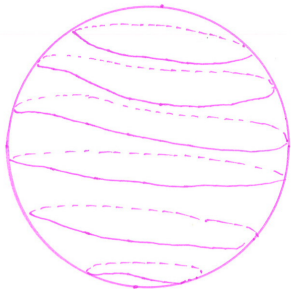
It is through the second variation that the geometry influences the behavior of minimal submanifolds. Positivity of the ambient curvature provides a focusing effect which strongly limits the behavior of stable minimal surfaces. (Think of the geodesics on the sphere, the great circles.) This general principle has many applications in geometry. One is to limit the topology of black holes in general relativity. We will discuss this more in tomorrow's lecture.

Min/Max Constructions

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The expected minimal surface should have Morse index at most p .

Existence and regularity of min/max hypersurfaces

The min/max construction of disk-type minimal surfaces was taken up by M. Morse and C. Tompkins in 1940, and more general variational theory was developed by F. Tomi and A. Tromba in the 1970s. Sacks and Uhlenbeck developed the variational theory for surfaces in curved spaces around 1980. In the context of integral currents the min/max theory was developed by F. Almgren in the 1960s.

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For $k = n - 1$ the partial regularity was proven around 1980 by the speaker and L. Simon completing work by J. Pitts who had handled low dimensional cases. A conclusion is that any compact Riemannian manifold has a minimal hypersurface which is smooth away from a singular set of Hausdorff dimension at most $n - 8$.

A Question about the Morse Index

In order to apply min/max theory to geometric situations it is important to understand the Morse index of solutions since it is through the second variation that the ambient geometry enters the theory.

Despite the regularity of the min/max hypersurface, the Morse index bound has not been proven. Recently, X. Zhou (arXiv:1210.2112) has solved this problem for metrics with positive Ricci curvature, showing that for simply connected M any basic min/max hypersurface has Morse index 1, while for general such M , any such hypersurface is either two sided with index 1 or a double copy of a stable one-sided hypersurface (this happens for RP^n).

Recent applications of the variational theory: Ricci flow

The proof of the three dimensional Poincaré Conjecture uses the variational theory for maps from S^2 into a simply connected three manifold M^3 . Given a metric g on M one can define the *width* of g by

$$w(g) = \min_{u(t)} \max_{t \in S^1} A(u(t))$$

where $u(t)$ denotes a continuous path of maps from S^2 to M which is homotopic to a standard path defining a sweepout of M by an S^1 family of maps.

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T. Colding and W. Minicozzi (following a suggestion of G. Perelman) showed that, for a solution $g(t)$ of the Ricci flow on M , the width $w(g(t))$ goes to zero in finite time. This implies the finiteness of the number of surgeries required in the Ricci flow to show that M is diffeomorphic to S^3 .

The Willmore conjecture I

Given a compact surface Σ in R^3 , the Willmore energy of Σ is defined by

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Thus we have $W(S^2) = 4\pi$.

This functional arises from physical considerations and is both scale and conformally invariant; that is, it is invariant under Möbius transformations. It is not difficult to show that for any closed surface Σ we have $W(\Sigma) \geq 4\pi$ with equality only for round spheres.

The Willmore Conjecture II

In 1965, T. Willmore made the conjecture that if Σ is homeomorphic to a torus, then we should have:

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The Willmore Conjecture II

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The torus achieving the minimum was conjectured to be the torus of revolution gotten by revolving a circle of radius 1 about an axis of distance $\sqrt{2}$ from the center of the circle; explicitly

$$(u, v) \rightarrow ((\sqrt{2} + \cos u) \cos v, (\sqrt{2} + \cos u) \sin v, \sin u)$$

for $0 \leq u, v \leq 2\pi$.

The Marques-Neves theorem

Fernando Marques and André Neves solved the Willmore Conjecture in the following strong form.

Theorem: If Σ is any closed immersed surface of positive genus in R^3 , then $W(\Sigma) \geq 2\pi^2$. Equality holds if and only if Σ is a conformal image of the torus described on the previous slide.

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The proof involves the use of min/max techniques for higher dimensional families of cycles.

A very recent application

In a very recent paper (arXiv:1311.6501) Marques and Neves have extended their techniques of min/max for higher dimensional families to prove the following:

Theorem: Every compact Riemannian manifold M^n for $3 \leq n \leq 7$ contains infinitely many smooth closed embedded minimal hypersurfaces.

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A conjecture of Yau is that every closed three manifold contains infinitely many closed embedded minimal surfaces.

Part 3: Minimal surfaces and eigenvalue problems

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To see this we can think of Σ as a submanifold of R^{n+1} and we have

$$\Delta(\vec{x}) = H_{R^{n+1}} = H_{S^n} + (H_{R^{n+1}} \cdot \vec{x})\vec{x} = -k\vec{x}$$

if and only if H_{S^n} is zero. Note that the component in the \vec{x} direction is the trace along Σ of the second fundamental form of S^n .

Minimal surfaces as eigenvalue extremals

In general the value k may not be the first eigenvalue, say $\lambda_p = k$. It turns out that the induced metric on a minimal surface in S^n is a stationary point for the variational problem associated with $\lambda_p(g)$ taken over metrics of fixed area on Σ .

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In the middle 1990s, N. Nadirashvili substantially generalized this to show that any smooth maximizing metric on a surface arises from a minimal immersion in S^n for some n by first eigenfunctions. He used this to determine the maximizing metric on a torus.

Minimal two spheres in S^n

The following theorem is due to H. Hopf and F. Almgren and uses a complex analytic construction called the Hopf differential which is constructed from the second fundamental form of a surface.

Theorem: Any minimal two sphere in S^3 is equatorial; that is a totally geodesic $S^2 \subseteq S^3$.

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The question of classifying minimal two spheres in higher dimensional spheres leads to a beautiful mathematical theory. First the isometric minimal immersions can be studied using representation theory. The ideas for the general case begin with E. Calabi and were developed by many authors in the context of integrable systems.

Free boundary minimal submanifolds in the ball

There is a class of minimal submanifolds with boundary which parallels the theory of minimal submanifolds of spheres. To motivate it we note that the cone over a minimal submanifold of the sphere is minimal in R^{n+1} , and if we look at the portion inside the unit ball, it meets the boundary sphere orthogonally.

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More generally we refer to a minimal submanifold of the ball which meets the boundary sphere orthogonally as a *free boundary solution*. The coordinate functions on such a Σ^k satisfy the condition that they are harmonic in the interior and satisfy the boundary condition

$$\frac{\partial x_i}{\partial \eta} = x_i$$

for $i = 1, 2, \dots, n$.

The eigenvalue problem

The coordinate functions on a free boundary minimal submanifold are called Steklov eigenfunctions; that is, they are eigenfunctions of the Dirichlet-Neumann map $L : C^\infty(\partial\Sigma) \rightarrow C^\infty(\partial\Sigma)$

$$Lu = \frac{\partial \hat{u}}{\partial \eta}$$

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In the third lecture we will explain why the free boundary minimal surfaces represent extremals over the space of metrics on a surface with boundary of a Steklov eigenvalue.

Free boundary minimal disks in B^n

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The following is a recent result with A. Fraser.

Theorem: Any free boundary minimal disk in B^n for $n \geq 4$ is a flat disk.

Outline of the proof

Assume that $\varphi : D \rightarrow B^n$ is a conformal parametrization of such a disk Σ . The conformality condition may be written $\varphi_z \cdot \varphi_z = 0$. The Hopf differential in complex form is given by

$$\Phi = \varphi_{zz}^\perp dz^2,$$

and it is a holomorphic quadratic differential with respect to the normal connection with values in the normal bundle.

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The free boundary condition implies that the normal vector valued $(0, 2)$ tensor Φ is real when applied to the boundary tangent. We then consider the complex valued quartic differential $\Phi \cdot \Phi$ and check that it is holomorphic and real along the boundary. It follows that $\Phi \cdot \Phi$ is identically 0 in D . From there we see that Φ is identically zero along ∂D and hence in D . This implies that the second fundamental form is zero and Σ is a flat disk.

Generalizations to CMC surfaces in space forms

The same basic argument can be used to prove the following.

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Corollary: A minimal S^2 in S^n which is invariant by reflection through a hyperplane is an equatorial S^2 (totally geodesic).