Extremal eigenvalue problems for surfaces

Richard Schoen

Stanford University

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Chen-Jung Hsu Lecture 3, Academia Sinica, ROC

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December 4, 2013

Plan of Lecture

The general lecture plan:

Part 1: Introduction: two classical theorems

Part 2: Surfaces with boundary

Part 3: Main theorems

Part 4: Outline of existence proof

Joint project with A. Fraser, arXiv:1209.3789

Part 1: Introduction: two classical theorems

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Given a smooth compact surface M, the choice of a Riemannian metric gives a Laplace operator which has a discrete set of eigenvalues $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots$

The following result was proven by J. Hersch in 1970.

Hersch Theorem: For any smooth metric g on $\mathbb{S}^2$ we have $\lambda_1(g)A(g) \leq 8\pi$ with equality if and only if g is a constant curvature metric.

Weinstock's Theorem

In 1954 R. Weinstock proved the following theorem.

Weinstock Theorem: Let M be a simply connected surface with boundary and g a smooth metric up to ∂M. We then have the inequality $\sigma_1(g)L(g) \leq 2\pi$ with equality if and only if (M, g) is equivalent to the unit disk in $\mathbb{R}^2$.

In the theorem, $L(g)$ denotes the length of ∂M with respect g and $\sigma_1(g)$ is the first nonzero Steklov eigenvalue of g.

Steklov eigenvalues I

$(M, \partial M)$ Riemannian manifold

Given a function $u \in C^\infty(\partial M)$, let $\hat{u}$ be the harmonic extension of u:

$$\begin{cases} \Delta_g \hat{u} = 0 & \text{on } \ M, \\ \hat{u} = u & \text{on } \partial M. \end{cases}$$

The Dirichlet-to-Neumann map is the map

$$L : C^\infty(\partial M) \to C^\infty(\partial M)$$

given by

$$Lu = \frac{\partial \hat{u}}{\partial \nu}.$$

(non-negative, self-adjoint operator with discrete spectrum)

Eigenvalues $\sigma_0, \sigma_1, \sigma_2, , \sigma_3, \dots$ (Steklov Eigenvalues)

Steklov eigenvalues II

Constant fcns are in the kernel of L
The lowest eigenvalue of L is zero, $\sigma_0 = 0$

The first nonzero eigenvalue σ_1 of L can be characterized variationally as:

$$\sigma_1 = \inf_{u \in C^1(\partial M),\ \int_{\partial M} u = 0} \frac{\int_M |\nabla \hat{u}|^2 \, dv_M}{\int_{\partial M} u^2 \, dv_{\partial M}}.$$

Example: B^m, $\quad \sigma_k = k, \quad k = 0, 1, 2, \ldots$
u homogeneous harmonic polynomial of degree k

$\sigma_1 = 1 \quad$ eigenspace $\quad x^1, \ldots, x^n$

Proof of Weinstock's theorem I

First recall the statement.

Weinstock Let M be a simply connected surface with boundary and g a smooth metric

$$\sigma_1(g)L(g) \leq 2\pi = \sigma_1(D)L(\partial D)$$

$=$ only if (M, g) is *equivalent* to a disk

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Proof.
RMT $\Longrightarrow \exists$ conformal diffeomorphism $\varphi : M \to D$
$\exists$ conformal $F : D \to D$ such that

$$\int_{\partial M} (F \circ \varphi) \, ds = 0$$

Proof of Weinstock's theorem II

i.e. WLOG, $\int_{\partial M} \varphi \, ds = 0$. Then, for $i = 1, 2$,

$$\sigma_1 \int_{\partial M} \varphi_i^2 \, ds \leq \int_M |\nabla \hat{\varphi}_i|^2 \, da$$
$$= \int_M |\nabla \varphi_i|^2 \, da$$

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$$\sigma_1 \int_{\partial M} \sum_{i=1}^{2} \varphi_i^2 \; ds \le \int_M \sum_{i=1}^{2} |\nabla \varphi_i|^2 \; da$$

$$\sigma_1 L(\partial M) \le 2\, A(\varphi(M)) = 2A(D) = 2\pi$$

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The proof of Hersch's theorem is similar with the conformal group of the disk replaced by that of the sphere.

Results for closed surfaces I

Is there an analogue to Hersh's theorem for other surfaces? In other words if I take a fixed compact surface M and consider all smooth metrics on M, which metrics maximize $\lambda_1 A$?

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• For RP^2 the constant curvature metric is the unique maximum by a result of P. Li and S. T. Yau from the 1980s.

• For T^2 the flat metric on the 60^0 rhombic torus is the unique maximum by a result of N. Nadirashvili from 1996.

Results for closed surfaces II

- For the Klein bottle the extremal metric is smooth and unique but not flat. This follows from work of Nadirashvili (1996 existence of maximizer), D. Jacobson, Nadirashvili, and I. Polterovich (2006 constructed the metric), and El Soufi, H. Giacomini, and M. Jazar (2006 proved it is unique).

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The case of the torus and the Klein bottle rely on a difficult existence theorem which was proved by Nadirashvili.

A recent paper by M. Karpukhin determines the extremal metric in the case of genus 2. The metric is the pullback from the metric on $\mathbb{S}^2$ under a degree 2 branched cover.

Part 2: Surfaces with boundary

Oriented surfaces with boundary are classified by their genus γ and the number of boundary components k. Such a surface is topologically equivalent to a closed surface of genus γ with k disjoint disks removed. Weinstock's theorem tells us that for $\gamma = 0$ and $k = 1$ the maximizing metric is the flat metric on the unit disk.

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The question we address is what are the maximizing metrics for more general surfaces with boundary. To be precise, given a smooth surface M with boundary, is there a smooth metric on M which maximizes $\sigma_1 L$? If so, can we describe the metric?

The structure of maximizing metrics

The following result shows that a metric which maximizes $\sigma_1 L$ arises from a free boundary minimal surface in a ball.

Theorem: Let M be a compact surface with nonempty boundary and assume that g is a metric on M for which $\sigma_1 L$ is maximized. The multiplicity of σ_1 is at least two, and there exists a proper branched minimal immersion $\varphi : M \to B^n$ for some $n \geq 2$ by first eigenfunctions which is a homothety on the boundary. The surface $\Sigma = \varphi(M)$ is a *free boundary minimal surface*. The extremal metric g is equal to a positive constant times the induced metric on the boundary.

Proof outline I

Step 1: Compute the first variation of the eigenvalue $\sigma(t)$ for a smoothly varying path of eigenfunctions u_t for a family of metrics g_t with fixed boundary length. This may be computed most easily by differentiating at $t = 0$

$$\int_M \|\nabla u_t\|_t^2 \, da_t = \sigma(t) \int_{\partial M} u_t^2 \, ds_t.$$

where g_t is a path of metrics with $g_0 = g$. If we denote derivatives at $t = 0$ with 'dots' we let $h = \dot{g}$, and the length constraint on the boundary translates to

$$\int_{\partial M} h(T, T) \, ds = 0$$

where T denotes the oriented unit tangent vector to ∂M.

Proof outline II

Denoting the derivative of σ at $t = 0$ by $\dot{\sigma}$,

$$\dot{\sigma} = -\int_M \langle \tau(u), h \rangle \; da - \int_{\partial M} u^2 h(T, T) \; ds$$

where $\tau(u)$ is the stress-energy tensor of u given by

$$\tau(u)_{ij} = \partial_i u \partial_j u - \frac{1}{2} \|\nabla u\|^2 g_{ij}$$

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Step 2: Assuming that $\sigma_1 L$ is maximized for g, show that for any variation h there exists an eigenfunction u (depending on h) with

$$Q_h(u) := \int_M \langle \tau(u), h \rangle \, da + \int_{\partial M} u^2 h(T, T) \, ds = 0.$$

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This is accomplished by using left and right hand derivatives to show that the quadratic form Q_h is indefinite and therefore has a null vector.

Proof outline III

Step 3: Use the Hahn-Banch theorem in an appropriate Hilbert space consisting of pairs (p, f) where p is a symmetric $(0, 2)$ tensor on M and f a function on ∂M to show that the pair $(0, 1)$ lies in the convex hull of the pairs $(\tau(u), u^2)$ for first eigenfunctions u. This tells us that there are positive $a_1, \ldots, a_n$ and eigenfunctions $u_1, \ldots, u_n$ so that

$$\sum a_i^2 \tau(u_i) = 0 \text{ on } M, \quad \sum a_i^2 u_i^2 = 1 \text{ on } \partial M.$$

It then follows that the map $\varphi = (a_1 u_1, \ldots, a_n u_n)$ is a (possibly branched) proper minimal immersion in the unit ball B^n. It can then be checked that φ is a homothety on ∂M.

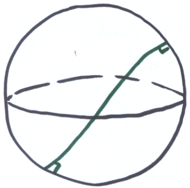
Free boundary minimal submanifolds I

A proper minimal submanifold Σ of the unit ball B^n which is orthogonal to the sphere at the boundary is called a *free boundary submanifold*. These are characterized by the condition that the coordinate functions are Steklov eigenfunctions with eigenvalue 1; that is, $\Delta x_i = 0$ in $\nabla_\eta x_i = x_i$.

The theorem shows that surfaces of this type arise as eigenvalue maximizers. Such submanifolds arise as min/max solutions for sweepouts of the ball by relative cycles and so have a natural variational interpretation.

Free boundary submanifolds II

$(M^k, \partial M^k) \longrightarrow (B^n, \partial B^n)$



M minimal, meeting ∂B^n orthogonally along ∂M

$\uparrow$ $\qquad\qquad\qquad\qquad$ $\uparrow$

$H = 0$ $\qquad\qquad\qquad$ $\eta = \vec{x}$

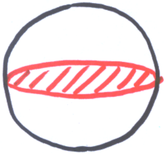
$M \subset \mathbb{R}^n$ minimal $\Longleftrightarrow \Delta_M x_i = 0 \quad i = 1, \ldots, n$

$(x_1, \ldots x_n$ are harmonic)

M meets ∂B^n orthogonally $\Longleftrightarrow \frac{\partial x_i}{\partial \eta} = x_i, \; i = 1, \ldots, n.$

Examples I

1) $M^k = D^k \subset B^n$ equatorial k-plane



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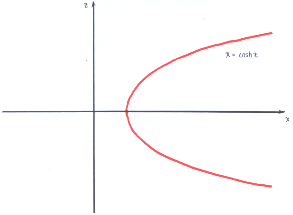
2) Critical Catenoid



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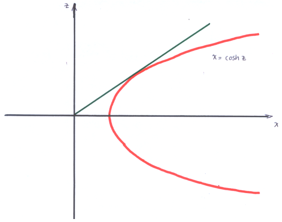

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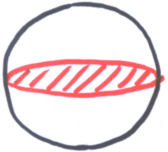

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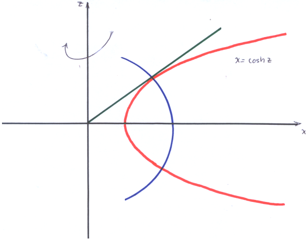

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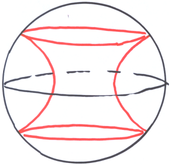

Examples II

3) We expect that there are arbitrarily high genus free boundary solutions with three boundary components in B^3 which converge to the union of the critical catenoid and a disk through the origin orthogonal to the axis.

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4) Critical Möbius Band

We think of the Möbius band M as $\mathbb{R} \times S^1$ with the identification $(t, \theta) \approx (-t, \theta + \pi)$. There is a minimal embedding of M into $\mathbb{R}^4$ given by

$$\varphi(t, \theta) = (2 \sinh t \cos \theta, 2 \sinh t \sin \theta, \cosh 2t \cos 2\theta, \cosh 2t \sin 2\theta)$$

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For a unique choice of T_0 the restriction of φ to $[-T_0, T_0] \times S^1$ defines an embedding into a ball by first Steklov eigenfunctions.

We may rescale the radius of the ball to 1 to get the *critical Möbius band*.

Explicitly T_0 is the unique positive solution of $\coth t = 2 \tanh 2t$.

Coarse upper bounds

The following result is a combination of bounds obtained with Fraser together with results of G. Kokarev.

Theorem: $(M^2, \partial M)$ *oriented Riemannian surface of genus* γ *with* k *boundary components. Then,*

$$\sigma_1 \, L(\partial M) \leq \min\{2\pi(\gamma + k), \; 8\pi[(\gamma + 3)/2]\}$$

The inequality is strict if $\gamma = 0$ *and* $k > 1$.

Note: $\gamma = 0$, $k = 1$ simply connected surface: Weinstock.

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Question: What is the sharp constant for annuli or other surfaces?

Part 3: Main theorems

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Theorem A: *Assume that Σ is a free boundary minimal annulus in B^n such that the coordinate functions are first eigenfunctions. Then $n = 3$ and Σ is the critical catenoid.*

There is a corresponding result for the Möbius band.

Theorem B: *Assume that Σ is a free boundary minimal Möbius band in B^n such that the coordinate functions are first eigenfunctions. Then $n = 4$ and Σ is the critical Möbius band.*

Main theorems on sharp bounds

Let

$$\sigma^*(\gamma, k) = \sup_g \sigma_1 L$$

where the supremum is over metrics on a surface of genus γ with k boundary components.

We know $\sigma^*(0,1) = 2\pi$. The next result identifies $\sigma^*(0,2)$.

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Theorem 1: *For any metric annulus M we have*

$$\sigma_1 L \leq (\sigma_1 L)_{cc}$$

with equality iff M is equivalent to the critical catenoid.
In particular,

$$\sigma^*(0,2) = (\sigma_1 L)_{cc} \approx 4\pi/1.2.$$

Theorem 2: *The sequence* $\sigma^*(0, k)$ *is strictly increasing in* k *and converges to* 4π *as* k *tends to infinity. For each* k *a maximizing metric is achieved by a free boundary minimal surface* Σ_k *in* B^3 *of area less than* 2π. *The limit of these minimal surfaces as* k *tends to infinity is a double disk, and for large* k, Σ_k *is approximately a pair of nearby parallel plane disks joined by* k *boundary bridges.*

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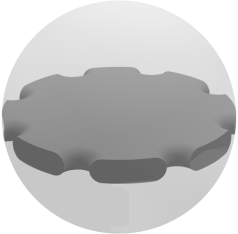
Here is a rough sketch of the surfaces for large k.



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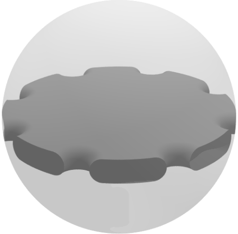

Corollary: *For every $k \geq 1$ there is an embedded minimal surface in B^3 of genus 0 with k boundary components satisfying the free boundary condition. Moreover these surfaces are embedded by first eigenfunctions.*

Existence of maximizers

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• Very roughly the proof involves first controlling the conformal structure of metrics near the supremum, and then controlling the metrics themselves.

Proof of Theorem 1

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• there exists a metric on the annulus with $\sigma_1 L = \sigma^*(0,2)$

• this maximizing metric arises from a free boundary minimal immersion of the annulus in the ball by first eigenfunctions

• by the uniqueness result this immersion is congruent to the critical catenoid

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• Σ_k does not contain the origin and is embedded and star shaped. This follows from the fact that the restrictions of the linear functions have no critical points on their zero set.

• The coarse upper bound implies that $A(\Sigma_k) \leq 4\pi$ and the star shaped property implies that each Σ_k is stable for variations which fix the boundary. Curvature estimates then imply uniform curvature bounds in the interior.

• There is a subsequence of the Σ_k which converges in a smooth topology to a smooth limiting minimal surface Σ_∞ possibly with multiplicity. This limit must have multiplicity since otherwise the limit would be a smooth free boundary solution, and the convergence would be smooth up to the boundary contradicting the fact that $k \to \infty$.

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• It follows from the star shaped condition that the origin lies in the limiting surface, the surface is a cone (hence a flat disk since it is smooth), and the multiplicity is 2.

• The limit of $A(\Sigma_k)$ is equal to $2A(\Sigma_\infty) = 2\pi$. It follows that $\sigma^*(0,k) = \sigma_1(\Sigma_k)L(\partial\Sigma_k) = 2A(\Sigma_k)$ converges to 4π as claimed.

The Möbius band

Finally we show that the critical Möbius band uniquely maximizes $\sigma_1 L$. After some calculation one can see that $(\sigma_1 L)_{cmb} = 6\sqrt{6}\pi$.

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Theorem: *For any metric on the Möbius band M we have*

$$\sigma_1 L \leq (\sigma_1 L)_{cmb}$$

with equality iff M is equivalent to the critical Möbius band.

The proof follows the same steps as for the critical catenoid.

- we show existence of a smooth maximizing metric on the Möbius band
- this then gives an immersion into B^n by first eigenfunctions
- by the uniqueness result, this immersion is congruent to the critical Möbius band

Statement of Theorem A

Theorem A: *Assume that* Σ *is a free boundary minimal annulus in* B^n *such that the coordinate functions are first eigenfunctions. Then* $n = 3$ *and* Σ *is the critical catenoid.*

Theorem A: Proof outline

A multiplicity bound implies that $n = 3$.

We may assume that Σ is parametrized by a conformal harmonic map φ from $M = [-T, T] \times S^1$ with coordinates (t, θ).

The vector field $X = \frac{\partial \varphi}{\partial \theta}$ is then a conformal Killing vector field along Σ.

Goal: Show that X coincides with a rotation vector field of $\mathbb{R}^3$.

The key step in doing this is to show that the three components of X are first eigenfunctions.

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Goal: Show that X coincides with a rotation vector field of $\mathbb{R}^3$.

The key step in doing this is to show that the three components of X are first eigenfunctions.

For functions or vector fields Y defined along Σ we consider the quadratic form Q defined by

$$Q(Y, Y) = \int_{\Sigma} \|\nabla Y\|^2 \, da - \int_{\partial \Sigma} \|Y\|^2 \, ds.$$

Assumption that $\sigma_1 = 1$ implies: if $\int_{\partial\Sigma} Y \, ds = 0$ then

$$Q(Y, Y) \geq 0$$

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It is easy to check that the vector field $X = \frac{\partial\varphi}{\partial\theta}$ is in the nullspace of Q. If it were true that $\int_{\partial\Sigma} X \, ds = 0$, then we could complete the argument.

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The quadratic form Q is also the second variation of $\frac{1}{2}E$ provided that Y is tangent to S^2 along $\partial\Sigma$.

Find a vector field Y such that $Q(Y, Y) \leq 0$ and with $\int_{\partial\Sigma}(X - Y) \, ds = 0$.

It would then follow that $Q(X - Y, X - Y) \leq 0$
and also, since $\int_{\partial\Sigma}(X - Y) \, ds = 0$, $Q(X - Y, X - Y) \geq 0$,

It would follow that the components of $X - Y$ are first e.f.

We are not able to find such vector fields directly, so we consider the second variation of area for normal variations.

Note that for free boundary solutions there is a natural Jacobi field given by $x \cdot \nu$. It is in the nullspace of the second variation form S given by:

$$S(\psi, \psi) = \int_{\Sigma} (\|\nabla \psi\|^2 - \|A\|^2 \psi^2)\ da - \int_{\partial \Sigma} \psi^2\ ds$$

where A denotes the second fundamental form of Σ and we are considering normal variations $\psi \nu$ where ν is the unit normal vector of Σ. Note that this variation is tangent to S^2 along the boundary.

We can show by a subtle argument that for any $v \in \mathbb{R}^3$

$$S(v \cdot \nu, v \cdot \nu) \leq 0$$

This is not sufficient for the eigenvalue problem because the normal deformation does not preserve the conformal structure of Σ in general.

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The way we get around this problem is to consider adding a tangential vector field Y^t to so that $Y = Y^t + \psi\nu$ preserves the conformal structure and is tangent to S^2 along $\partial\Sigma$. This involves solving a Cauchy-Riemann equation with boundary condition to determine Y^t.

This problem is generally not solvable, but has a 1 dimensional obstruction for its solvability (because Σ is an annulus). We then get existence for ψ in a three dimensional subspace of the span of $\nu_1, \nu_2, \nu_3, x \cdot \nu$. We can then arrange the resulting conformal vector field Y to satisfy the boundary integral condition and we have

$$Q(Y, Y) = S(\psi, \psi) = S(v \cdot \nu, v \cdot \nu) \leq 0.$$

Part 4: Outline of existence proof

We are left with proving Theorem 2 which is the existence and regularity result under that assumption that the conformal structures can be controlled for metrics near the maximum. This is the problem of controlling the boundary measures under the assumption that we have a controlled conformal background.

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One might approach this variational problem by completing the space of boundary measures in the weak* topology to allow singular measures. Given a measure μ and a smooth constant curvature metric g_0 we can define $\sigma_1(g_0, \mu)$ by

$$\sigma_1(g_0, \mu) = \inf\{\frac{E(u)}{\int_{\partial M} u^2 \, d\mu} : \ u \in H^1(M) \cap C^0(\bar{M}), \ \int_{\partial M} u \, d\mu = 0\}.$$

It is then easy to see that $\sigma_1(g_0, \mu)$ is upper-semicontinuous in μ, so we can construct a maximizing probability measure.

Preliminary remarks

Thus if we define $\sigma^{**}(\gamma, k)$ by

$$\sigma^{**}(\gamma, k) = \sup\{\sigma_1(g_0, \mu) : \ \mu(\partial M) = 1\},$$

then under the assumption of Theorem 2 we can find a maximizing measure μ.

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We then have to deal with the problem of showing that μ is a smooth measure. This seems to be a very difficult problem; in fact, it is not clear that $\sigma^{**}(\gamma, k) = \sigma^*(\gamma, k)$. It could conceivably happen that the eigenvalue could be made strictly larger by allowing μ to be singular.

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We take a different approach which involves finding a special maximizing sequence of smooth measures which satisfy a regularized problem, and showing that these converge weak* to a smooth measure.

The regularized problem

Since we have a background conformal metric which we can control, we use it to regularize our boundary measures. We let $K_t(x, y)$ denote the heat kernel with respect the boundary arclength measure induced by g_0. Thus $K_t(x, y)$ depends on $x, y \in \partial M$, but it is zero if x and y lie in different boundary components.

Given $\epsilon > 0$ and a measure μ on ∂M we let μ_ϵ be the smooth measure $\mu_\epsilon = \lambda_\epsilon \; ds_0$ where

$$\lambda_\epsilon(x) = \int_{\partial M} K_\epsilon(x, y) \; d\mu(y).$$

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We can then pose (and solve) the regularized problem of realizing σ_ϵ^* given by

$$\sigma_\epsilon^* = \sup\{\sigma_1(g_0, \mu_\epsilon) : \; \mu, \; g_0\}.$$

Under the hypothesis of Theorem 2 we can assume that the g_0 lie in a compact set of metrics, and because of the regularization we can construct a maximizing probability measure $\mu^{(\epsilon)}$.

The convergence question

Since the conformal structures are controlled we make the simplifying assumption that the conformal structure is fixed given by g_0. We hope to show that there is a sequence ϵ_i tending to 0 so that the weak* limit μ of the $\mu_i = \mu^{(\epsilon_i)}$ is a smooth measure. If we can do this, then it follows from upper-semicontinuity of $\sigma_1(g_0, \mu)$ with respect to μ that

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Since μ is a smooth measure we can write $\mu = \lambda\ ds_0$ for a smooth positive function λ on ∂M. We can then extend λ as a smooth positive function in M which we also call λ, and then it follows that the smooth metric $g = \lambda^2 g_0$ is a smooth maximizing metric. This will complete the proof of Theorem 2.

Three basic eigenvalue results

The convergence proof uses three basic results about the eigenvalue problem. These are important for proving the convergence.

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1) Multiplicity Bound: The multiplicity of σ_1 is bounded by $4\gamma + 3$.

2) Sparsity of Nodal Set: There is a constant N depending only on γ and k so that the number of points on ∂M at which a first eigenfunction vanishes is bounded by N.

3) Persistence of Nodal Set: If there is a point $x \in \partial M$ for which $u(x) = 0$, then there is a fixed radius $r_0 > 0$ so that for each $r \leq r_0$, the nodal set intersects the circle of radius r centered at x.

Maximizers for the regularized problem

Assume g_0 is a fixed constant curvature metric and $\epsilon > 0$ is fixed. Let μ be a maximizing measure for g_0 and ϵ; that is, $\sigma_1(g_0, \mu, \epsilon) = \sigma_\epsilon^*$. There are independent first eigenfunctions $u_1, \ldots, u_n$ such that the map $u = (u_1, \ldots, u_n)$ is a harmonic map into $\mathbb{R}^n$ with the following properties $(|u|^2)_\epsilon = 1$ and $\nabla_T (|u|^2)_\epsilon = 0$ for μ-almost all points of ∂M where T is the unit tangent vector with respect to g_0.

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Furthermore, if g_0 is chosen so that σ_ϵ^* is maximized over g_0, then in addition to the above conditions on the map u, we may assume that for any h of compact support the function

$$F(t) = \int_M |\nabla_t u|^2 \, da_t - \sigma_\epsilon^* \int_{\partial M} |u|^2 \, d(\mu)_{t,\epsilon}$$

has a critical point at $t = 0$ where we have $g_t = g + th$, and the quantities are computed with respect to g_t. Note that $(\mu)_{t,\epsilon}$ denotes the solution of the heat equation with initial data μ for the canonical metric in the conformal class of g_t.

Bounds on the $\sqrt{\epsilon}$ scale

The heat kernel regularization roughly corresponds to averaging over intervals of length $\sqrt{\epsilon}$, so it is reasonable to expect that on that scale we have u_ϵ close to u, and both are uniformly well behaved.

Lemma: There is a continuous increasing function $\omega(\epsilon)$ with $\omega(0) = 0$ with the following properties: For any point x in the support of $\mu^{(\epsilon)}$ and for any $y \in I_{\omega(\epsilon)^{-1}\sqrt{\epsilon}}(x)$ we have $|u(x) - u(y)| \leq \omega(\epsilon)$, $|u_\epsilon(x) - u(x)| \leq \omega(\epsilon)$, and $||u(x)| - 1| \leq \omega(\epsilon)$. Furthermore we have $|u| \leq c$ for all $x \in \partial M$ for a fixed constant c independent of ϵ.

Weak limits

Now suppose that ϵ_i is a sequence tending to 0 with μ_i being a corresponding maximizing measure and u_i the corresponding harmonic map. There are two types of weak limits we can take. First we may assume that μ_i converges weak* to a measure μ. We may also assume that u_i converges weakly in H^1 to a limit u (a harmonic map to R^n for some $n \geq 1$).

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Since $|u_i|$ is uniformly bounded, for $j = 1, \ldots, n$ we may assume the signed measures u_i^j (the jth component of u_i) converge weak* to a limiting measure which is absolutely continuously with respect to μ. This limiting measure may be written $\hat{u}^j\mu$, so we have an $L^\infty(\partial M, \mu)$ map $\hat{u}$ defined for μ-almost every point of ∂M .

Conditions satisfied by u

Since u_i is harmonic with $\nabla_\eta u_i = \sigma_i u_i$ on ∂M, we may take the weak limit to conclude that u is harmonic and satisfies the boundary condition $\nabla_\eta u \; ds_0 = \sigma^* \hat{u}\mu$ in the sense that for any smooth map φ from M to R^n we have

$$\int_M \langle \nabla u, \nabla \varphi \rangle \; da_0 = \sigma^* \int_{\partial M} \langle \hat{u}, \varphi \rangle \; d\mu.$$

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$$\int_M \langle \nabla u, \nabla \varphi \rangle \; da_0 = \sigma^* \int_{\partial M} \langle \hat{u}, \varphi \rangle \; d\mu.$$

It also follows from the conditions on maximizers for the regularized problem that u is a conformal map.

Nontriviality of weak limits

While μ is a probability measure, there is no guarantee that either u or $\hat{u}$ is nontrivial. From the equation satisfied for u we see that u will be nontrivial if $\hat{u}$ is.

Lemma: There is a positive constant $c = c(n)$ such that $|\hat{u}| \geq c$ for μ almost all points of ∂M.

Nontriviality of weak limits

While μ is a probability measure, there is no guarantee that either u or $\hat{u}$ is nontrivial. From the equation satisfied for u we see that u will be nontrivial if $\hat{u}$ is.

<u>Lemma</u>: There is a positive constant $c = c(n)$ such that $|\hat{u}| \geq c$ for μ almost all points of ∂M.

To prove this we use the properties of the nodal set. Since for $j = 1, \ldots, n$ there are a bounded number of zeros of u_i^j on ∂M, we can choose a subsequence so that these zeros converge, and hence there are a finite number of intervals I of ∂M whose union is ∂M so that for i large each of the functions u_i^j is nonzero at all points of a slightly smaller interval J. Now for μ_i-almost all points of ∂M we have $|u_i|$ arbitrarily close to 1. It follows that the image of J under u_i lies in an octant of R^n arbitrarily close to the unit sphere. Since the convex hull of this set lies a fixed distance from the origin, we may conclude that the same is true for the weak limit $\hat{u}$.

Uniform equicontinuity of the angle

The sequence of maps u_i is uniformly equicontinuous on the support of μ_i, and on any interval I of ∂M for which $|u_i|$ is bounded from below by a positive constant, the sequence $u_i/|u_i|$ is uniformly equicontinuous.

• It follows that $\hat{u}$ is continuous and $|\hat{u}| = 1$ on the support of μ. The support of μ is all of ∂M since u is a nontrivial minimal immersion and so cannot have vanishing normal derivative on an open interval of ∂M.

• It also follows that there is a μ-measurable function a with values in $[0, 1]$ such that $u = a\hat{u}$ at μ-almost all points of ∂M.

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• It also follows that there is a μ-measurable function a with values in $[0, 1]$ such that $u = a\hat{u}$ at μ-almost all points of ∂M.

The proof uses the persistence of the nodal set in a crucial way.

Regularity of the weak limit

The equation satisfied by u (the Laplace equation) implies the equations in spherical coordinates (ρ, ξ)

$$\Delta \xi^i + \sum_{\alpha=1}^{2} \sum_{j,k=1}^{n-1} \Gamma^i_{jk}(\xi) \frac{\partial \xi^j}{\partial x^\alpha} \frac{\partial \xi^k}{\partial x^\alpha} + \sum_{\alpha=1}^{2} \frac{\partial \xi^i}{\partial x^\alpha} \frac{\partial \rho}{\partial x^\alpha} = 0$$

where Γ^i_{jk} are the Christoffel symbols for the standard metric on $\mathbb{S}^{n-1}$. The conformality condition on u implies

$$(\frac{\partial \rho}{\partial z})^2 = -\rho^2 \langle \frac{\partial \xi}{\partial z}, \frac{\partial \xi}{\partial z} \rangle$$

where the inner product is taken with respect the spherical metric and $z = x^1 + \sqrt{-1}x^2$.

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where the inner product is taken with respect the spherical metric and $z = x^1 + \sqrt{-1} x^2$.

We can show from these equations that since ξ is continuous, ξ is $C^{1,\alpha}$ up to ∂M. It then follows that u is $C^{1,\alpha}$ up to ∂M.

Completion of the proof

Since $\nabla_\eta u$ is parallel to u and u is conformal, it follows that the tangential derivative $\nabla_T u$ is orthogonal to u. Thus we have $\nabla_T |u|^2 = 0$, and we see that $|u|$ is a positive constant on each component of ∂M. Thus locally the image of u is a free boundary surface and the higher regularity follows from standard theory. We have from the boundary condition that μ is a positive constant times the smooth measure $|\nabla_\eta u|\ ds_0$, and so μ is a smooth measure. As observed earlier, this completes the proof because of the upper-semicontinuity of σ_1.