MIRROR SYMMETRY AND REAL FORMS

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MIRROR SYMMETRY

- symplectic geometry: A-model

- complex geometry: B-model

- $+$ Calabi-Yau

BRANES

- symplectic geometry

- A-brane = Lagrangian submanifold ....+ flat bundle

BRANES

- symplectic geometry

- A-brane = Lagrangian submanifold ....+ flat bundle

- complex geometry

- B-brane = complex submanifold...+ holomorphic bundle

HYPERKÄHLER MANIFOLDS

- complex structures I, J, K
- $I^2 = J^2 = K^2 = IJK = -1$
- symplectic forms $\omega_1, \omega_2, \omega_3$
- $\omega_2 + i\omega_3$ I-holomorphic symplectic etc.
- $\Rightarrow$ Calabi-Yau

- complex structures I, J, K

- symplectic structures $\omega_1, \omega_2, \omega_3$

- (B, A, A)-brane: cx wrt I, totally real wrt J, K

- (B, B, B)-brane: HK submanifold + hyperholomorphic bundle

- hyperholomorphic bundle

- $\sim$ holomorphic wrt I, J, K

- $\Leftrightarrow$ connection A, curvature $F_A \in \Omega^{1,1}$

- .... for I, J, K

- for a hyperkähler manifold....

- M has mirror hyperkähler $\widehat{M}$

- mirror symmetry: (B,A,A)-brane $\leftrightarrow$ (B,B,B)-brane

- $\Rightarrow$ (B,A,A)-brane $\rightarrow$ hyperkähler submanifold

  $+$ hyperholomorphic bundle

HIGGS BUNDLES

- algebraic curve Σ
- holomorphic G^c-principal bundle
- section $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K)$ = Higgs field
- $+$ stability condition

- $\Rightarrow$ reduction to maximal compact G
- $\Rightarrow$ G-connection A
- $F_A + [\Phi, \Phi^*] = 0$
- moduli space hyperkähler

- complex structure I: moduli space of (stable) pairs (A,Φ)

  $G=U(n)$ vector bundle V, $\Phi\in H^0(\Sigma,\mathrm{End}\,V\otimes K)$

- complex structure J: flat G^c-connection

  $\nabla_A+\Phi+\Phi^*$ (representations $\pi_1(\Sigma)\to G^c$)

- complex structure K: flat G^c-connection

  $\nabla_A+i\Phi-i\Phi^*$

FLAT CONNECTIONS FOR A REAL GROUP

- $G^r \subset G^c$ real form (e.g. $SL(n,\mathbf{R}) \subset SL(n,\mathbf{C})$)

- $\mathrm{Hom}(\pi_1(\Sigma), G^r) \subset \mathrm{Hom}(\pi_1(\Sigma), G^c)$

- What is the corresponding Higgs bundle?

REAL FORM G^r

- $K \subset G^r$ maximal compact
- principal K^c-bundle
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$
- Higgs field $\Phi \in H^0(\Sigma, \mathfrak{m} \otimes K)$
- holonomy of $\nabla + \Phi + \Phi^* \in G^r$

EXAMPLE $G^r = SL(n, \mathbf{R})$

- orthogonal vector bundle V

- $\Lambda^n V \cong \mathcal{O}$

- $\Phi = \Phi^T \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$

- involution on hyperkähler moduli space $\mathcal{M}$
- holomorphic wrt I
- anti-holomorphic wrt J and K
- fixed-point set (B, A, A)-brane

EXAMPLE $G^r = SL(n, \mathbf{R})$

- holomorphic involution $V \mapsto V^*$
- .... and $\Phi \mapsto \Phi^T$
- anti-holomorphic involution on Hom$(\pi_1, SL(n, \mathbf{C}))$
- Fixed point set may have many components

EXAMPLE $G^r = SL(2,\mathbf{R})$

- maximal compact $SO(2)$, Chern class $c \in H^2(\Sigma, \mathbf{Z}) = \mathbf{Z}$

- $|c| \leq 2g - 2$ (Milnor-Wood)

- $|c| < 2g - 2$ determines a connected component

- $|c| = 2g - 2 \Rightarrow 2^{2g}$ connected components

MIRROR SYMMETRY

THE FIBRATION

- hyperkähler moduli space $\mathcal{M}(G)$

  $\dim_{\mathbf{R}} = 4(g-1)\dim G$

- principal G^c-bundle, $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K)$

- invariant polynomials $p_1, \ldots, p_\ell$ on $\mathfrak{g}$

  $p_m(\Phi) \in H^0(\Sigma, K^{d_m})$

- fibration $\mathcal{M}^{2k}(G) \to \mathbf{C}^k$

- integrable system

- generic fibre abelian variety A

- $G^c = GL(n, \mathbf{C})$ $\det(x - \Phi) = 0$ spectral curve S

- fibre = $\mathrm{Jac}(S)$

- spectral curve S: $\det(x - \Phi) = 0$
- curve in the cotangent bundle $\pi : K \to \Sigma$
- $\pi : S \to \Sigma$ n-fold cover

- $V = \pi_* L$
- $\Phi = \pi_*(L \xrightarrow{x} L \otimes \pi^* K)$
- $x \in \pi^* K$ canonical section

- $G^c = SL(n, \mathbf{C})$

- $L = U \otimes \pi^* K^{(n-1)/2}$, $\deg U = 0$

- $\mathrm{Nm} : \mathrm{Pic}^0(S) \to \mathrm{Pic}^0(\Sigma)$

- $U \in \mathrm{P}(S, \Sigma) =$ Prym variety $=$ kernel

SYZ MIRROR SYMMETRY

- special Lagrangian fibration

- mirror = dual fibration

- $\mathcal{M}(G)$ is mirror to $\mathcal{M}({}^L G)$

R.Donagi & T.Pantev, *Langlands duality for Hitchin systems,* Invent. math. **189** (2012), 653–735.

- dual $A^\vee$ of an abelian variety A = moduli space of degree zero line bundles on A

- dual $A^\vee$ of an abelian variety $A =$ moduli space of degree zero line bundles on A

- $B \subset A$ subvariety

- line bundles trivial on B defines

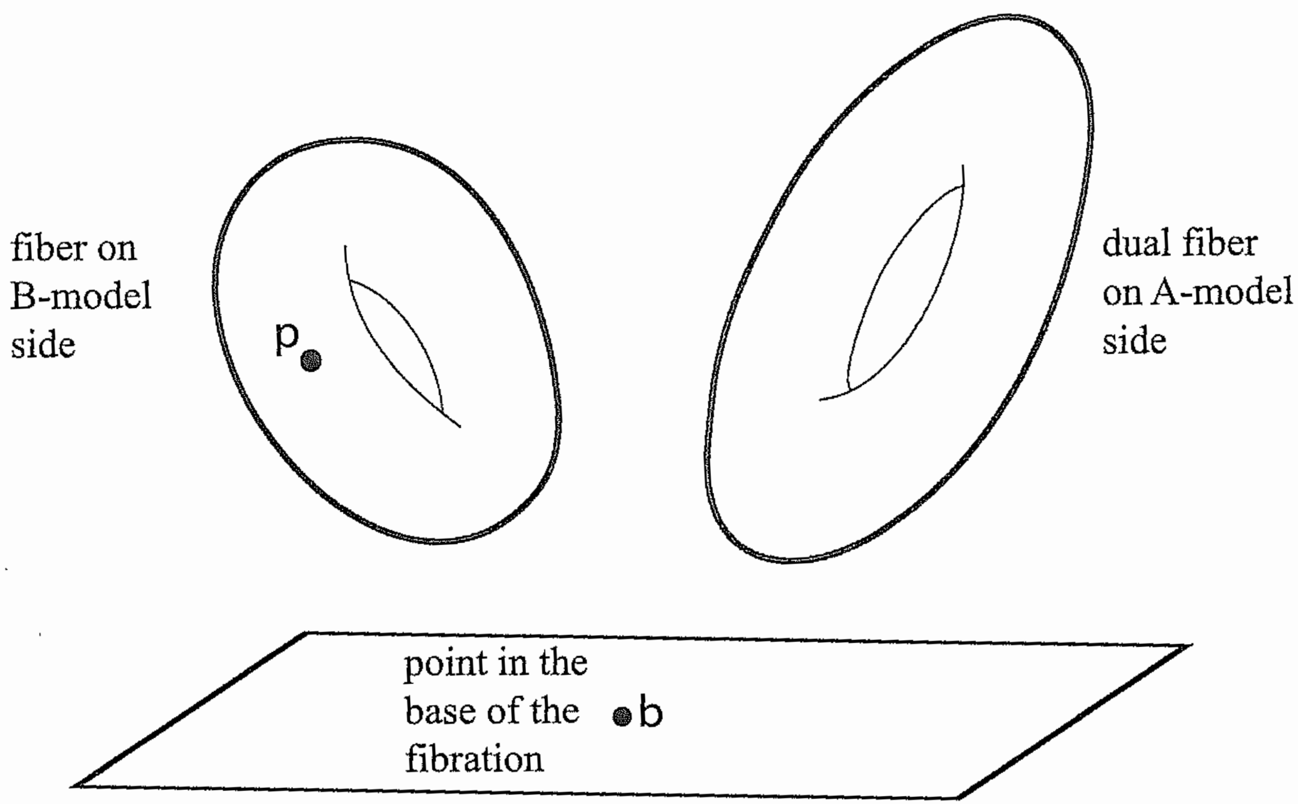
- .. $B^0 \subset A^\vee$

- fibre $A \subset \mathcal{M}(G)$ maps to a point in base $\mathbf{C}^k$

- A is I-holomorphic, but $(\omega_2 + i\omega_3)|_A = 0 \Rightarrow$ Lagrangian wrt ω_2, ω_3 = (B,A,A)-brane

- mirror = point = the trivial bundle over A

the dual torus, which is the fiber of $M(X,G)$ over the same point b (the rig
torus on the picture, on the A-model side). The dual A-brane on $M(X,G)$
are looking for will be the A-brane "smeared" over this dual torus. It will
the same dual brane as the one we obtain under the mirror symmetry betwe
these two tori.



TWO ISSUES

1. Given G^r find a hyperkähler submanifold of $\mathcal{M}({}^L G)$
2. For each component of the moduli space of flat G^r-connections, find a hyperholomorphic bundle on this submanifold.

THE HYPERKÄHLER SUBMANIFOLD

	$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{g}$	$\check{\mathfrak{g}}$	$\check{\mathfrak{h}}$	Remarks
AI	$\mathfrak{sl}_n(\mathbb{R})$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	split
AII	$\mathfrak{su}^*(2n)$	$\mathfrak{sl}_{2n}(\mathbb{C})$	$\mathfrak{sl}_{2n}(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	
AIII/AIV	$\mathfrak{su}(p,q)$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sp}_p(\mathbb{C})$	$p \leq q$ $p+q=n$ quasi-split if $q=p$ or $q=p+1$
BI/BII	$\mathfrak{so}(p,q)$	$\mathfrak{so}_{2n+1}(\mathbb{C})$	$\mathfrak{sp}_n(\mathbb{C})$	$\mathfrak{sp}_p(\mathbb{C})$	$p<q$ $p+q=2n+1$ split if $q=p+1$
CI	$\mathfrak{sp}_n(\mathbb{R})$	$\mathfrak{sp}_n(\mathbb{C})$	$\mathfrak{so}_{2n+1}(\mathbb{C})$	$\mathfrak{so}_{2n+1}(\mathbb{C})$	split
CII	$\mathfrak{sp}(p,q)$	$\mathfrak{sp}_n(\mathbb{C})$	$\mathfrak{so}_{2n+1}(\mathbb{C})$	$\mathfrak{sp}_p(\mathbb{C})$	$p \leq q$ $p+q=n$
DI/DII	$\mathfrak{so}(n,n)$	$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{so}_{2n}(\mathbb{C})$	split
	$\mathfrak{so}(p,q)$	$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{so}_{2p+1}(\mathbb{C})$	$p<q$ $p+q=2n$ quasi-split if $q=p+2$
DIII	$\mathfrak{so}^*(2n)$	$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{sp}_p(\mathbb{C})$	$p=[n/2]$
EI	$\mathfrak{e}_{6(6)}$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C})$	split
EII	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{f}_4(\mathbb{C})$	quasi-split
EIII	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{so}_5(\mathbb{C})$	
EIV	$\mathfrak{e}_{6(-26)}$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{sl}_3(\mathbb{C})$	
EV	$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{e}_7(\mathbb{C})$	split
EVI	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{f}_4(\mathbb{C})$	
EVII	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{sp}_3(\mathbb{C})$	
EVIII	$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_8(\mathbb{C})$	$\mathfrak{e}_8(\mathbb{C})$	$\mathfrak{e}_8(\mathbb{C})$	split
EIX	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_8(\mathbb{C})$	$\mathfrak{e}_8(\mathbb{C})$	$\mathfrak{f}_4(\mathbb{C})$	
FI	$\mathfrak{f}_{4(4)}$	$\mathfrak{f}_4(\mathbb{C})$	$\mathfrak{f}_4(\mathbb{C})$	$\mathfrak{f}_4(\mathbb{C})$	split
FII	$\mathfrak{f}_{4(-20)}$	$\mathfrak{f}_4(\mathbb{C})$	$\mathfrak{f}_4(\mathbb{C})$	$\mathfrak{sl}_2(\mathbb{C})$	
G	$\mathfrak{g}_{2(2)}$	$\mathfrak{g}_2(\mathbb{C})$	$\mathfrak{g}_2(\mathbb{C})$	$\mathfrak{g}_2(\mathbb{C})$	split

TABLE 1. Associated Lie algebras $\check{\mathfrak{h}}$ for non-compact real Lie algebras $\mathfrak{g}_{\mathbb{R}}$ with simple complexifications $\mathfrak{g}$. Notation following É. Cartan, and [Hel78].

real forms of G^c

complex subgroups of ${}^L G^c$

	$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{g}$	$\check{\mathfrak{g}}$	$\check{\mathfrak{h}}$	Remarks
AI	$\mathfrak{sl}_n(\mathbb{R})$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	split
AII	$\mathfrak{su}^*(2n)$	$\mathfrak{sl}_{2n}(\mathbb{C})$	$\mathfrak{sl}_{2n}(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	
AIII/AIV	$\mathfrak{su}(p,q)$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sp}_p(\mathbb{C})$	$p \leq q$ $p+q=n$ quasi-split if $q=p$ or $q=p+1$
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D.Nadler, *Perverse sheaves on real loop Grassmannians*, Invent. Math. **159** (2005) 1–73

- $G^r \subset G^c$

- $\Rightarrow \hat{H} \subset {}^L G^c$

Conjecture The mirror of the moduli space of flat G^r-bundles is supported on the Higgs bundle moduli space $\mathcal{M}(\hat{H}) \subset \mathcal{M}({}^L G)$.

- split real form $G^r \subset G^c$

- e.g. $SL(n, \mathbf{R}) \subset SL(n, \mathbf{C})$

- $\widehat{H} = {}^L G^c$

- Hom$(\pi_1, G^r)/G^r \subset \mathcal{M}(G^c) \to \mathbf{C}^k$

- generic fibre finite $= A[2]$

- line bundles on A trivial on finite set $= \hat{A}$

- Hom$(\pi_1, G^r)/G^r \subset \mathcal{M}(G^c) \to \mathbf{C}^k$

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- $\dim \mathrm{Hom}(\pi_1, G^r)/G^r = \dim \mathcal{M}(G^c)/2$

- .. maps surjectively onto the base

L.Schaposnik, *Spectral data for G-Higgs bundles,* arXiv:1301.198

THE CASE $G^r = U(m, m)$

- maximal compact $U(m) \times U(m)$

- bundle $V = V_+ \oplus V_-$ Higgs field $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$

- characteristic class $c_1(V_+) \in H^2(\Sigma, \mathbf{Z})$

- $\Rightarrow$ different topological components

L.Schaposnik, *Spectral data for G-Higgs bundles,* arXiv:1301.1981

- spectral curve $\det(x-\Phi) = x^{2m} + a_1 x^{2m-2} + \ldots + a_m$

- involution $\sigma(x) = -x$ on S

- $L = U\pi^* K^{(2m-1)/2}$, $U \in \mathrm{Jac}(S)$

- $\sigma^* U \cong U$

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- $L = U\pi^* K^{(2m-1)/2}$, $U \in \mathrm{Jac}(S)$

- $\sigma^* U \cong U$

- $c_1(V_+) = m(g-1) - k$

  $2k$ = no of fixed points where action is -1

- spectral curve $\det(x-\Phi) = x^{2m} + a_1 x^{2m-2} + \ldots + a_m$
- base of fibration = even degree differentials
- mirror supported on a hyperkähler submanifold which projects to this

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- moduli space of $Sp(2m, \mathbf{C})$ Higgs bundles.

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SPECTRAL CURVE FOR $Sp(2m)$

- rank $2m$ bundle E, skew form $\langle\ ,\ \rangle$

  Higgs field Φ: $\langle \Phi v, w\rangle + \langle v, \Phi w\rangle = 0$

- eigenvectors v_i, v_j, $\langle v_i, v_j\rangle = 0$ unless $\lambda_i = -\lambda_j$

- characteristic polynomial $x^{2m} + a_2 x^{2m-2} + \ldots + a_{2m}$

- $U(m,m)$ fibre: disconnected

- $\sigma^* U \cong U$

- if U_1, U_2 have the same action at fixed points then....

- $U_2 \cong U_1 \otimes q^* L$ where $q : S \to S/\sigma = \bar{S}$

- each component $\cong$ Jac($\bar{S}$)

- dual of q^* : Jac($\bar{S}$) $\rightarrow$ Jac(S) is ..

- ... Nm : Jac(S) $\rightarrow$ Jac($\bar{S}$)

- kernel = Prym variety Prym($S, \bar{S}$)

- = fibre of $Sp(2m, \mathbf{C})$ fibration

THE CASE $G^r = SU^*(2m)$

- $SU^*(2m) = SL(m, \mathbf{H})$

- maximal compact $Sp(m)$

- V symplectic, $\Phi^T = \Phi$ (symplectic transpose)

- $\Rightarrow \det(x - \Phi) = (x^m + a_2 x^{m-1} + \ldots + a_m)^2$

- call S: $x^m + a_2 x^{m-1} + \ldots + a_m = 0$ the spectral curve

- $V = \pi_* E$ rank 2 bundle

- fibre $\cap$ Hom$(\pi_1, SU^*(2m))/SU^(2m)$ = moduli space of stable rank 2 bundles on S

- fibre no longer an abelian variety – many algebraic components

- What is the rest of the fibre?

- characteristic polynomial $(x^m + a_2 x^{m-1} + \ldots + a_m)^2 = p(x)^2$

- $V = \pi_* E$ rank 2 bundle $\sim$ minimal polynomial $= p(x)$

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- general case minimal polynomial $= p(x)^2$

- extension of Higgs bundles

$$0 \to (W_1, \Phi_1) \to (V, \Phi) \to (W_2, \Phi_2) \to 0$$

where S is the spectral curve for (W_1, Φ_1) and (W_2, Φ_2)

$$W_1 = \pi_* L_1, W_2 = \pi_* L_2$$

- extensions classified by hypercohomology $\mathbf{H}^1$ of

$$\mathcal{O}(\mathrm{Hom}(W_2,W_1)) \stackrel{\Psi}{\to} \mathcal{O}(\mathrm{Hom}(W_2,W_1)\otimes K)$$

$$\Psi(A) = A\Phi_2 - \Phi_1 A$$

- smooth $S \Rightarrow$ sheaves $\ker\Psi, \mathrm{coker}\,\Psi$ vector bundles

- $\mathrm{coker}\,\Psi \cong \ker\Psi \otimes K^m$

Note: If $W_1 = W_2$, hypercohomology $\mathbf{H}^1$ = tangent space to $\mathcal{M}(GL(m))$

- spectral sequence $\Rightarrow$

$$0 \to H^1(\ker \Psi) \to \mathbf{H}^1 \xrightarrow{\delta} H^0(\operatorname{coker} \Psi) \to 0$$

- $s \in H^0(\pi^{-1}(U), L_1^* L_2)$

$$s : H^0(\pi^{-1}(U), L_1) \to H^0(\pi^{-1}(U), L_2)$$

$$\psi : H^0(U, W_1) \to H^0(U, W_2)$$

- spectral sequence $\Rightarrow$

$$0 \to H^1(\ker \Psi) \to \mathbf{H}^1 \xrightarrow{\delta} H^0(\operatorname{coker} \Psi) \to 0$$

- $s \in H^0(\pi^{-1}(U), L_1^* L_2)$

$$s : H^0(\pi^{-1}(U), L_1) \to H^0(\pi^{-1}(U), L_2)$$
$$\psi : H^0(U, W_1) \to H^0(U, W_2)$$

- $sx = xs \Rightarrow$ defines $\psi \in H^0(U, \ker \Psi)$

- $\ker \Psi \cong \pi_*(L_1^* L_2)$

$$0 \to H^1(\ker \Psi) \to \mathbf{H}^1 \xrightarrow{\delta} H^0(\text{coker}\, \Psi) \to 0$$

- if $\delta(a) = 0$

  $\bar{\partial}$-operator on extension =

$$\bar{\partial} + \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

  where α intertwines Φ_1, Φ_2

- $\Rightarrow$ $\Phi = \begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix}$

- $\Rightarrow$ minimal polynomial $p(x)$

- non-zero section of $H^0(\Sigma, \mathrm{coker}\,\Psi) \cong H^0(S, L_1^* L_2 K^m)$

- $\sim$ symmetric product $S^{[d]}$

- $+$ extension in $H^1(S, \ker\Psi) \cong H^1(S, L_1^* L_2)$

- $\deg L_1^* L_2 K^m = d \sim$ different (algebraic) components

- non-zero section of $H^0(\Sigma, \mathrm{coker}\,\Psi) \cong H^0(S, L_1^* L_2 K^m)$

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- $\deg L_1^* L_2 K^m = d \sim$ different (algebraic) components

- $L \in \mathsf{Jac}(S)$: $(L_1, L_2) \mapsto (LL_1, LL_2)$

- $\mathsf{Nm}\, L_1 + \mathsf{Nm}\, L_2 = 0 \Rightarrow 2\,\mathsf{Nm}\, L = 0$

- abelian variety $A = \{[L] \in \mathsf{Jac}(S) : L^2 \in \mathsf{Prym}(S)\}$

- moduli space $\mathcal{N}$ of rank 2 stable bundles on S: Pic $= \mathbf{Z}$

- line bundle on fibre, trivial on $\mathcal{N}$: degree zero

- degree zero line bundles on $A = A^{\vee}$

- mirror hyperkähler submanifold $\sim SL(m, \mathbf{C})$ Higgs bundles

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HYPERHOLOMORPHIC BUNDLES

HYPERHOLOMORPHIC BUNDLE

- Higgs bundle equations: dimensional reduction of ASDYM

- ASD connection $A_1 dx_1 + A_2 dx_2 + \phi_1 dx_3 + \phi_2 dx_4$

- $D^* = \nabla_1 + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \nabla_2 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi_1 + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \phi_2$

- Dirac operator $D^* = \begin{pmatrix} \bar{\partial}_A & \Phi \\ \Phi^* & \partial_A \end{pmatrix}$

$$D^* = \begin{pmatrix} \bar{\partial}_A & \Phi \\ \Phi^* & \partial_A \end{pmatrix} : \begin{pmatrix} V \otimes K \\ V \otimes \bar{K} \end{pmatrix} \to \begin{pmatrix} V \otimes K\bar{K} \\ V \otimes K\bar{K} \end{pmatrix}$$

- $D^*D \sim -\nabla_1^2 - \nabla_2^2 - \phi_1^2 - \phi_2^2 \Rightarrow \ker D = 0$
- index theorem $\Rightarrow \dim\ker D^* = (2g-2)\,\mathrm{rk}\,V$
- $\mathcal{L}^2$ connection is hyperholomorphic

- complex structure I

- $$\begin{array}{ccc} \Omega^{0,p}(V) & \stackrel{\Phi}{\to} & \Omega^{0,p}(V \otimes K) \\ \bar{\partial}\downarrow & & \bar{\partial}\downarrow \\ \Omega^{0,p+1}(V) & \stackrel{\Phi}{\to} & \Omega^{0,p+1}(V \otimes K) \end{array}$$

- total differential $\bar{\partial} \pm \Phi$

- Hodge theory: $\ker D^* \cong$ hypercohomology $\mathbf{H}^1$

THE TANGENT BUNDLE

- tangent space = hypercohomology $\mathbf{H}^1$ of

$$\mathcal{O}(\mathfrak{g}) \stackrel{[\Phi,\ ]}{\rightarrow} \mathcal{O}(\mathfrak{g} \otimes K)$$

- = de Rham cohomology $H^1(\Sigma, \mathfrak{g})$ of local system

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THE DIRAC BUNDLE

hypercohomology $\mathbf{H}^1$ of

- $$\mathcal{O}(V) \stackrel{\Phi}{\rightarrow} \mathcal{O}(V \otimes K)$$

- = de Rham cohomology $H^1(\Sigma, V)$ of local system

- $G^r = U(m, m)$

- different components $\Rightarrow$ different hyperholomorphic bundles

- ... on the $Sp(2m, \mathrm{C})$-moduli space

THE HYPERHOLOMORPHIC BUNDLE

- V = rank m bundle + symplectic form

- hypercohomology $\mathbf{H}^1$ of $\mathcal{O}(V) \stackrel{\Phi}{\to} \mathcal{O}(V \otimes K)$ defines a hyperholomorphic bundle $\mathbf{V}$

- $\det(x - \Phi) = x^{2m} + a_1 x^{2m-2} + \ldots + a_m$

- $\det \Phi = 0$ if $a_m = 0$, zero set Z of section of K^{2m}

- $$\mathbf{H}^1 \cong \bigoplus_{z \in Z} \operatorname{coker} \Phi_z.$$

- $\Phi : V \to V \otimes K$ generically an isomorphism

- $0 \to \mathbf{H}^1 \to H^0(\mathrm{coker}\,\Phi) \to 0$

- $\mathrm{coker}\,\Phi$ supported on $\det \Phi = 0$

- spectral curve S: $x^{2m} + a_1 x^{2m-2} + \ldots + a_m = 0$

- $Z \cong$ fixed point set $x = 0$ of $\sigma(x) = -x$

- on S: $\mathbf{H}^1 \cong \bigoplus_{z \in Z} (L\pi^* K)_z$

- component of $U(m,m)$ moduli space $\sim 2k$-element subset of Z, where σ acts as -1

- $$\Lambda^{2k}\mathbf{V} \cong \bigoplus_{\{z_1,\ldots,z_{2k}\}\subset Z} (L\pi^*K)_{z_1}(L\pi^*K)_{z_2}\ldots(L\pi^*K)_{z_{2k}}$$

- .... sum over $2k$-element subsets

- **Claim**: $\Lambda^{2k}\mathbf{V}$ is the required hyperholomorphic bundle

- no universal bundle on $Sp(2m,\mathbf{C})$ moduli space

- gerbe, class in $H^2(\mathcal{M},\mathbf{Z}_2)$

- no universal bundle on $Sp(2m, \mathbf{C})$ moduli space

- gerbe, class in $H^2(\mathcal{M}, \mathbf{Z}_2)$

- local $\mathbf{V}_\alpha$, $\mathbf{V}_\beta \cong \mathbf{V}_\alpha \otimes L_{\alpha\beta}$

- $L^2_{\alpha\beta}$ trivial so $\Lambda^{2k}\mathbf{V}_\beta \cong \Lambda^{2k}\mathbf{V}_\alpha$

REMARKS:

- Dirac operator quaternionic, V quaternionic ...

- $\Rightarrow$ hyperholomorphic $SO(2m)$-connection on $\mathbf{V}$

- $\Rightarrow \Lambda^{2m-2k}\mathbf{V} \cong \Lambda^{2k}\mathbf{V}$

- $\Lambda^0\mathbf{V}$ = maximal value of $c_1(V_+)$

THE CASE $G^r = SU(m,m)$

- Prym varieties $\text{Prym}(\bar{S}, \Sigma) \subset \text{Prym}(S, \Sigma)$

- $\text{P}(S, \bar{S})/\text{P}(S, \bar{S}) \cap \text{Jac}(\Sigma) \to \text{Jac}(S)/\,\text{Jac}(\Sigma) \to \text{Jac}(\bar{S})/\,\text{Jac}(\Sigma)$

- Prym varieties $\mathsf{Prym}(\bar{S},\mathbf{\Sigma})\subset\mathsf{Prym}(S,\mathbf{\Sigma})$

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- $\mathsf{P}(S,\bar{S})\cap\mathsf{Jac}(\mathbf{\Sigma})=\{U\in\mathsf{Jac}(\mathbf{\Sigma}):\sigma^*\pi^*U\cong\pi^*U^*\}=H^1(\mathbf{\Sigma},\mathbf{Z}_2)$

- mirror = quotient of $Sp(2m,\mathbf{C})$ moduli space by $H^1(\mathbf{\Sigma},\mathbf{Z}_2)$

- quotient = $PSp(2m, \mathbf{C})$ moduli space

- **Problem**: $\mathbf{V}$ doesn't descend

- ???

- canonical section of fibration $\Rightarrow PSL(2, \mathbf{R})$-moduli space

- mirror $=$ trivial holomorphic bundle on $SL(2, \mathbf{C})$-moduli space

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- mirror = trivial holomorphic bundle on $SL(2,\mathbf{C})$-moduli space

- 2^{2g} canonical sections of $SL(2,\mathbf{C})$-moduli space:

  $V = K^{1/2} \oplus K^{-1/2}$

- mirror = 2^{2g} trivializations of the gerbe on $PSL(2,\mathbf{C})$ moduli space (Frenkel-Witten)