HIGGS BUNDLES AND DIFFEOMORPHISM GROUPS

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HIGHER TEICHMÜLLER SPACES

- compact surface Σ

- $\mathrm{Hom}(\pi_1(\Sigma), SL(n,\mathbf{R}))/SL(n,\mathbf{R})$

- there is a component $\cong \mathbf{R}^N$ $(N = 2(n^2-1)(g-1))$

NJH, *Lie groups and Teichmüller space*, Topology **31** (1992), 449 – 473.

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- $n = 2$ Teichmüller space

- What geometry does $n > 2$ parametrize?

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What geometry does $n = \infty$ parametrize?

THE GROUP $SU(\infty)$

- SDiff(S^2) = group of symplectic diffeomorphisms of S^2

- $df = i_X\omega$ Hamiltonian vector fields

- Lie algebra = $C^\infty(S^2)/const.$

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- $PSU(2) \subset$ SDiff(S^2)

- spherical harmonics $C^\infty/const. = \mathbf{3} + \mathbf{5} + \mathbf{7} + \ldots$

- $SU(2) \to SU(n)$ irreducible representation

- $\mathfrak{su}(n) = \mathbf{3} + \mathbf{5} + \mathbf{7} + \ldots + (\mathbf{2n} - \mathbf{1})$

- $SU(\infty) \stackrel{\text{def}}{=} \mathrm{SDiff}(S^2)$

- Poisson bracket $\neq$ Lie bracket (except on $SU(2)$)

PROPERTIES

- invariant metric

$$(f,g) = \int_{S^2} fg\omega$$

- invariant polynomials

$$p_n(f) = \int_{S^2} f^n\omega$$

- $\sim$ compact Lie group G

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- ... but no complexification G^c

THE GROUP $SO(\infty)$

- $S^2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$

- involution $\sigma(x_1, x_2, x_3) = (x_1, x_2, -x_3)$

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- $SO(\infty) = \{f \in \mathsf{SDiff}(S^2) : f\sigma = \sigma f\}$
- Lie algebra $= \mathfrak{h} =$ odd functions on S^2
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad \mathfrak{m} =$ even functions

- $SO(\infty)$ preserves the fixed point set $x_3 = 0$

- homomorphism $SO(\infty) \to \text{Diff}(S^1)$

HIGGS BUNDLES

- compact Riemann surface Σ, compact group G

- principal G-bundle P + connection A

- Higgs field $\Phi \in \Omega^{1,0}(\Sigma, \mathfrak{g}^c)$

- equations

$$F_A + [\Phi, \Phi^*] = 0, \quad \bar{\partial}_A \Phi = 0$$

- G^c-connection $\nabla_A + \Phi + \Phi^*$
- equations $\Rightarrow$ flat G^c-connection
- conversely, given a reductive representation $\pi_1(\Sigma) \to G^c$
- a harmonic section of $\tilde{\Sigma} \times_{\pi_1} G^c/G$
- ... defines a solution to the Higgs bundle equations

- G^c connection $\nabla_A + \Phi + \Phi^*$

- equations $\Rightarrow$ flat connection

- real form $G^r \subset G^c$, max compact H

- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$

- flat G^r connection if A reduces to H and $\Phi \in H^0(\Sigma, \mathfrak{m} \otimes K)$

EXAMPLE: $G = SU(2), G^c = SL(2,\mathbf{C}), G^r = SL(2,\mathbf{R})$

- $V = K^{-1/2} \oplus K^{1/2}$

  $$\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- $U(1)$-connection A on $K^{1/2}$

- $F_A + [\Phi, \Phi^*] = 0 \Rightarrow K = -1$ (Gaussian curvature

- uniformization: $\pi_1(\Sigma) \to SL(2,\mathbf{R})$

- $V = K^{(n-1)/2} \oplus K^{(n-3)/2} \oplus \ldots \oplus K^{-(n-1)/2}$

- $\Phi =$ companion matrix

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & a_{n-1} & \ldots & \ldots & 0 \end{pmatrix}$$

- $a_i \in H^0(\boldsymbol{\Sigma}, K^i)$

- higher Teichmüller space for $SL(n, \mathbf{R})$

$SU(\infty)$ HIGGS BUNDLES

$SU(\infty)$-CONNECTION

- 2-sphere bundle $p : M^4 \to \Sigma$
- non-vanishing section ω_F of $\Lambda^2 T_F^*$
- horizontal subbundle $H \subset TM$
- such that for each horizontal lift of a vector field X on M...
- .... $\mathcal{L}_X \omega_F = 0$

$SO(\infty)$-CONNECTION

- 2-sphere bundle $p : M^4 \to \Sigma$
- ... etc.
- $+$ anti-symplectic involution on fibres
- fixed point circle bundle
- Diff(S^1)-connection

$SU(\infty)$-HIGGS FIELD

- Locally $\phi_1 dx_1 + \phi_2 dx_2$
- ϕ_i functions on Σ with values in $C^\infty(S^2)$
- $\sim$ functions on M
- $\Phi = (\phi_1 dx_1 + \phi_2 dx_2)^{1,0}$ section of p^*K on M

$SL(\infty, \mathbf{R})$

- $SO(\infty)$ connection
- Higgs field $\phi_1 dx_1 + \phi_2 dx_2$, ϕ_1, ϕ_2 even functions
- Poisson bracket $\{\phi_1, \phi_2\}$ odd
- $\Rightarrow$ vanishes on circle bundle

- connection: $$\frac{\partial}{\partial x} + A_1, \quad \frac{\partial}{\partial y} + A_2$$

  A_1, A_2 Hamiltonian vector fields on S^2 depending on x, y

- Higgs field:

  Φ_1, Φ_2 Hamiltonian vector fields on S^2 depending on x, y

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- $$\left[\frac{\partial}{\partial x} + A_1 - i\Phi_2, \frac{\partial}{\partial y} + A_2 + i\Phi_1\right] = 0$$

  (*cf.* $\nabla_A + \Phi + \Phi^*$ flat)

- complex vector fields

$$X_1 = \frac{\partial}{\partial x} + A_1 - i\Phi_2, \quad X_2 = \frac{\partial}{\partial y} + A_2 + i\Phi_1$$

$$[X_1, X_2] = 0$$

- $\Rightarrow$ integrable complex structure

- as long as $X_1, \bar{X}_1, X_2, \bar{X}_2$ are linearly independent

- iff Hamiltonian vector fields Φ_1, Φ_2 are linearly independent

- Hamiltonian functions ϕ_1, ϕ_2 for vector fields Φ_1, Φ_2

- linear dependence where $\{\phi_1, \phi_2\} = 0$ (Poisson bracket)

- defines a hypersurface N^3

- for $SL(\infty, \mathbf{R})$, $N^3 \sim$ circle bundle

$$\left[\frac{\partial}{\partial x}+A_1+i\left(\frac{\partial}{\partial y}+A_2\right),\Phi_1+i\Phi_2\right]=0$$

$$\left[\frac{\partial}{\partial x}+A_1+i\Phi_2,\frac{\partial}{\partial y}+A_2-i\Phi_1\right]=0$$

- two more complex structures $\Rightarrow$ hypercomplex manifold

- symplectic $\Rightarrow$ hyperkähler

THE CANONICAL MODEL

THE CANONICAL MODEL

- $V = K^{-1/2} \oplus K^{1/2}$

$$\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- $SU(2) \subset SU(\infty)$
- real form $SO(2) \subset SO(\infty)$, $SL(2, \mathbf{R}) \subset SL(\infty, \mathbf{R})$
- $\Rightarrow$ hyperkähler manifold with $N =$ circle bundle

Theorem (Feix, Kaledin) Given a real analytic Kähler metric on M there is a unique S^1-invariant hyperkähler extension to a neighbourhood of the zero section in T^*M.

- $\omega_2 + i\omega_3 =$ canonical complex symplectic form

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- $M = S^2$ complete metric (Eguchi-Hanson)

- $M = \Sigma$ surface of genus $g > 1$ incomplete (complete $\Rightarrow$ polynomial growth in π_1)

J.D.Gegenberg & A.Das, *Stationary Riemannian space-times with self-dual curvature,* Gen. Relativity Gravitation **16** (1984) 817–829.

H.Pedersen & B.Nielsen, *On some Euclidean Einstein metrics,* Lett.Math.Phys. **12** (1986) 277–282.

S.K.Donaldson, *Moment maps in differential geometry,* Surv. Differ. Geom., **8** Int. Press, Somerville, MA, 2003 171-189.

- Kähler form ω_1 on fibre $\quad i\dfrac{dwd\bar{w}}{4(1-|w|^2)^{1/2}}$

$$i\frac{dwd\bar{w}}{4(1-|w|^2)^{1/2}} = \frac{dx_1 \wedge dx_2}{2(1-x_1^2-x_2^2)^{1/2}} = \frac{dx_1 \wedge dx_2}{2x_3}$$

- well-defined on S^2

- $\omega_1, \omega_2, \omega_3$ well-defined on $M^4 \stackrel{S^2}{\to} \Sigma$

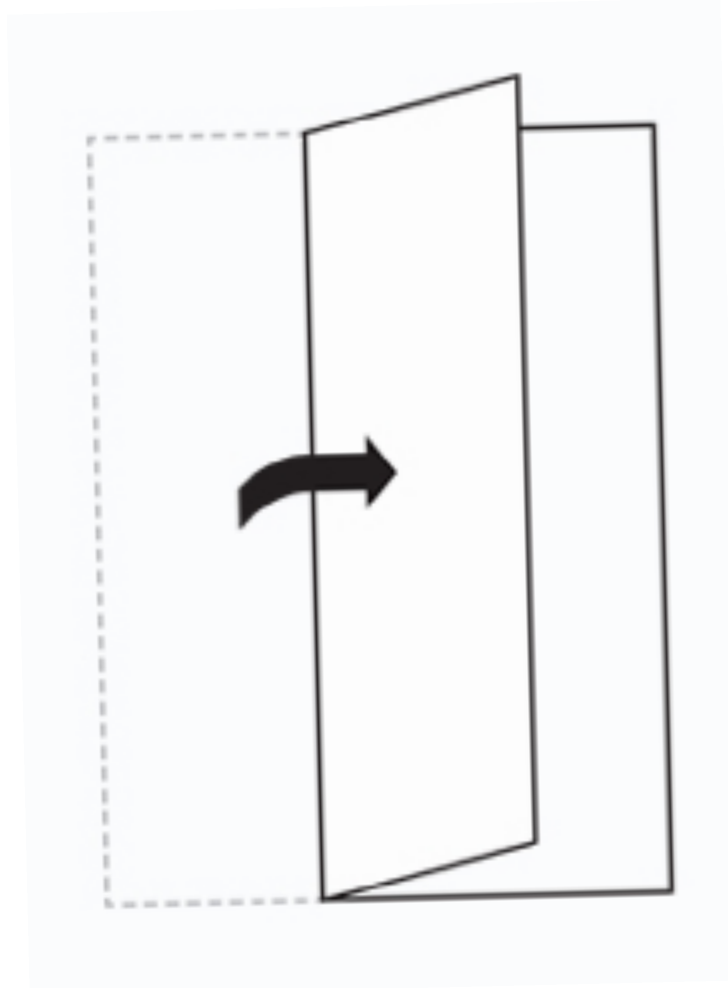
- take Σ with hyperbolic metric

- $T^*\Sigma \subset \mathbf{P}(K \oplus 1)$

- the hyperkähler extension is defined on the unit disc bundle in $T^*\Sigma$

- .... and extends to a folded hyperkähler metric on the S^2-bundle $\mathbf{P}(K \oplus 1)$

FOLDING



- $f : \mathbf{R}^2 \to \mathbf{R}^2 \qquad f(x, y) = (x^2, y)$

- $f^*(dx \wedge dy) = 2x dx \wedge dy$

- symplectic manifold M^{2m}: closed 2-form ω, $\omega^m \neq 0$

- folded symplectic manifold:M^{2m}: closed 2-form ω

- ... $\omega^m = 0$ defines a smooth hypersurface N^{2m-1}

- ... and $\omega|_N$ has maximal rank

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- ... and $\omega|_N$ has maximal rank

- normal form $\quad xdx \wedge dy + \sum_1^{m-1} du_i \wedge dv_i$

Theorem: Any compact oriented 4-manifold admits a folded Kähler structure.

R I Baykur, *Kähler decompositions of 4-manifolds*, AGT **6** (2006) 1239–1265.

(symplectic geometry of Stein surfaces)

- $M^4 = M^+ \cup N^3 \cup M^-$

- Kähler metric $\pm$ definite on $M^\pm$

HYPERKÄHLER GEOMETRY

4D HYPERKÄHLER MANIFOLD

- metric g, complex structures I, J, K

- Kähler forms $\omega_1, \omega_2, \omega_3$

- $\omega_1^2 = \omega_2^2 = \omega_3^2$

- $\omega_1\omega_2 = \omega_2\omega_3 = \omega_3\omega_1 = 0$

- metric $g = \omega_1\omega_2^{-1}\omega_3$

FOLDED HYPERKÄHLER

- closed 2- forms $\omega_1, \omega_2, \omega_3$

- $\omega_1^2 = \omega_2^2 = \omega_3^2$

- $\omega_1\omega_2 = \omega_2\omega_3 = \omega_3\omega_1 = 0$

- $\omega_1^2 = 0$ defines a smooth hypersurface N^3

α-PLANES

$$\omega_1 = dx \wedge \varphi + x d\varphi$$

$$\omega_2 = x dx \wedge \alpha_1 + \beta_1 \wedge \varphi$$

$$\omega_3 = x dx \wedge \alpha_2 + \beta_2 \wedge \varphi$$

β-PLANES

$$\omega_1 = x dx \wedge \alpha_0 + \beta_0 \wedge \gamma_0$$

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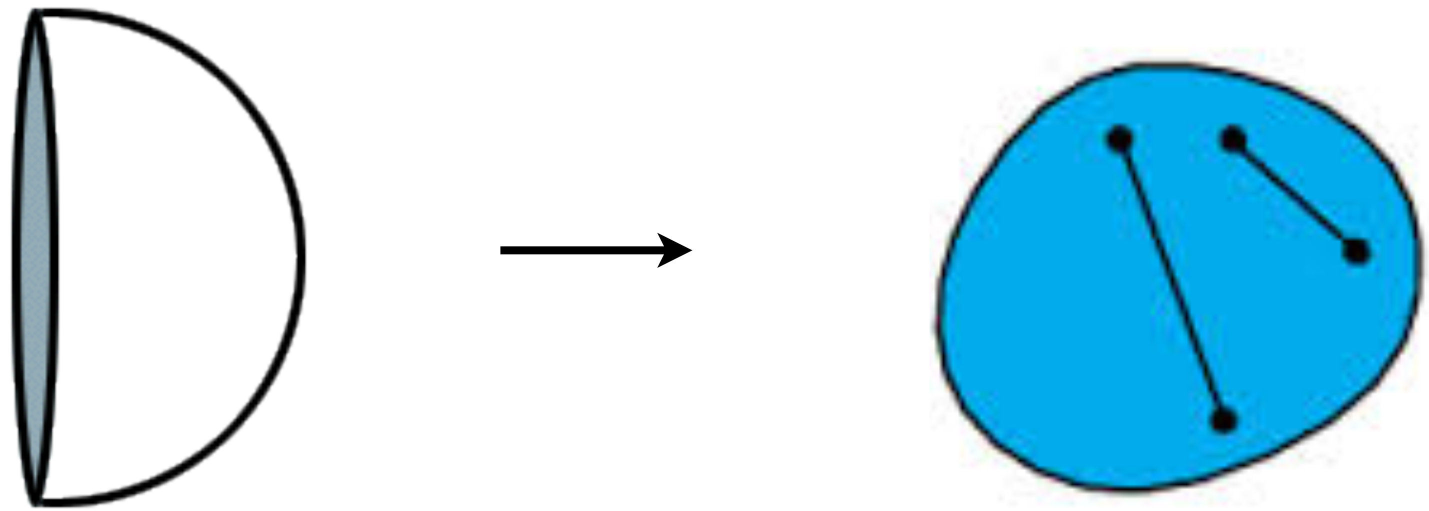
GEOMETRY OF THE FOLD

- Higgs field Φ section of p^*K

- map $f : M^4 \to T^*\Sigma$

- Higgs field $\Phi = f^*(wdz)$ canonical 1-form

- $\omega_2 + i\omega_3 = f^*(dw \wedge dz)$



- f maps the fold to a (non-quadratic) circle bundle in $T^*\Sigma$

- Finsler geometry = circle bundle in $T\Sigma$

- Legendre transform $\Rightarrow T^*\Sigma$

- hyperkähler forms near $\{\phi_1, \phi_2\} = x = 0$

  $\omega_1 = dx \wedge \varphi + x d\varphi$

  $\omega_2 = x dx \wedge \alpha_1 + \beta_1 \wedge \varphi$

  $\omega_3 = x dx \wedge \alpha_2 + \beta_2 \wedge \varphi$

- on the fold N^3:

  closed 2-forms $\beta_1 \wedge \varphi, \beta_2 \wedge \varphi$

- on $f(N)$ restrictions of real and imaginary parts of $dw \wedge dz$

- on $f(N)$, annihilator of β_1, φ = one-dimensional foliation

- $\sim$ Hamiltonian flow

- β_2 restricts to a parameter on the integral curve

- annihilator of φ = horizontal subspaces = Diff(S^1)-connection

EXAMPLE: CANONICAL MODEL

- N = unit (cotangent) circle bundle for hyperbolic metric
- foliation = geodesic flow
- $\beta_2 = ds$ length

THE EQUATIONS

$SL(\infty, \mathbf{C})$

- the space of Kähler potentials on S^2
- $\omega = \omega_0 + i\partial\bar{\partial}\phi$
- formally a symmetric space
- $\sim SL(\infty, \mathbf{C})/SU(\infty)$

S.K.Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics* in Northern California Symplectic Geometry Seminar, AMS (1999) 13–33.

- $M^4 \to \Sigma$ fibre disc

- hyperkähler $\Rightarrow$ flat connection on the bundle of (folded) Kähler potentials over Σ

- hyperkähler form ω_3 restricted to fibre = section

- harmonic section

S.K.Donaldson, *Nahm's equations and free boundary problems* in "The Many Facets of Geometry", (eds. O. Garcia-Prada et al) OUP (2010)

A QUESTION

- for each group $SL(n, \mathbf{R})$ there is a distinguished component of $\mathrm{Hom}(\pi_1(\Sigma), SL(n, \mathbf{R}))/SL(n, \mathbf{R})$

- ... Teichmüller space for $n = 2$

- .... $\cong \bigoplus_{m=2}^{n} H^0(\Sigma, K^m)$ using Higgs bundles

- Is there an analogue for $SL(\infty, \mathbf{R})$ and does it parametrize generalized geodesic structures?

EVIDENCE 1

- circle action $\Phi \mapsto e^{i\theta}\Phi$

- for higher Teichmüller space unique fixed point

- ... = uniformizing representation

$$\pi_1(\Sigma) \to SL(2, \mathbf{R}) \stackrel{S^m}{\to} SL(m+1, \mathbf{R})$$

Is the canonical model the only S^1-invariant folded hyperkähler manifold of this type?

- S^1-invariance = $SU(\infty)$ Toda equation
- locally $(e^u)_{tt} + u_{xx} + u_{yy} = 0$

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- 2-dim Toda lattice

$$e^{v_{n+1}} - 2e^{v_n} + e^{v_{n-1}} + v_{xx} + v_{yy} = 0$$

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$$e^{v_{n+1}} - 2e^{v_n} + e^{v_{n-1}} + v_{xx} + v_{yy} = 0$$

- circle action $\partial/\partial\tau$

- metric

$$ds^2 = u_t(4e^u dz d\bar{z} + dt^2) + u_t^{-1}[2d\tau + i(u_z dz - u_{\bar{z}} d\bar{z})]^2$$

- Kähler form

$$i\omega_1 = 4u_t e^u dz \wedge d\bar{z} + i dt \wedge [2d\tau + i(u_z dz - u_{\bar{z}} d\bar{z})]$$

- holomorphic symplectic form

$$\omega_2 + i\omega_3 = d(w dz)$$

where $w = 4e^{u/2+i\tau}$

- M = disc bundle $\subset T^*\Sigma$
- z local complex coordinate on Σ, w fibre coordinate
- wdz = canonical (holomorphic) 1-form on $T^*\Sigma$
- holomorphic symplectic form = $d(wdz)$
- circle action $w \mapsto e^{i\tau}w$

- metric + Killing vector field $X = \partial/\partial\tau$

- Kähler form

$$i\omega_1 = 4u_t e^u dz \wedge d\bar{z} + i dt \wedge [2d\tau + i(u_z dz - u_{\bar{z}} d\bar{z})]$$

- $i_X \omega_1 = -2dt$, $t \sim$ moment map

- $t = const =$ circle bundle over Σ

- Kähler quotient $= \Sigma$ with metric $4u_t e^u (dx^2 + dy^2)$

- another metric $w\bar{w}dzd\bar{z} = 16e^u(dx^2 + dy^2)$ on Σ
- metric $g(x, y, t)$ on Σ for $t > 0$
- $g_t = 4\times$ quotient metric.

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- $g_t = 4\times$ quotient metric.

  - conformal structure same for all t

  - Toda equation:

$$\frac{\partial^2 g}{\partial t^2} = Kg$$

- boundary conditions

- $t = 0$: zero section $w = 0$ of $T^*\Sigma$

- $\Rightarrow g(0) = 0$

- $t = 1 =$ fold $g_t(0) = 0$

- $g_{tt} = Kg \Rightarrow$ volume is quadratic in t $\qquad (g = f(x,t)g_0(x))$

- rescale g to constant volume metric h

- put $h = fg_H$, g_H=hyperbolic metric

$$t(2-t)f_{tt} + 4(1-t)f_t = \Delta_H \log f.$$

- boundary conditions + maximum $\Rightarrow f = const.$

EVIDENCE 2

- deformations of fixed point of circle action

- for higher Teichmüller space $\sim$ holomorphic sections of $K^2, K^3,$

- Also for $SL(\infty, \mathbf{R})$?

- finite dimensions: invariant polynomials $\operatorname{tr}\Phi^m \in H^0(\Sigma, K^m)$

- Higher Teichmüller space = section

- finite dimensions: invariant polynomials $\mathrm{tr}\,\Phi^m \in H^0(\Sigma, K^m)$

- Higher Teichmüller space = section

- $SL(\infty, \mathbf{R})$: $f : D \subset M^4 \to T^*\Sigma$

- $$\left(\int_{f(D)} w^m f_*\omega\right) dz^m \in H^0(\Sigma, K^m)$$

- $\theta = w dz$ = canonical 1-form, $\omega = dw \wedge dz$

- $\alpha = a dz^k$ holomorphic section of K^k

- $h = dz d\bar{z}/y^2$ hyperbolic metric

complex vector field

$$X^c = \omega^{-1} \alpha h^{-(k-1)} \bar{\theta}^{k-1} = a y^{2(k-1)} \bar{w}^{k-1} \frac{\partial}{\partial w}$$

- X= real part of X^c
- $\Rightarrow$ the closed 2-forms $\mathcal{L}_X\omega_i$ are anti-self-dual
- first order deformation $\dot{\omega}_1 = 0, \dot{\omega}_2 = \mathcal{L}_X\omega_2, \dot{\omega}_3 = \mathcal{L}_X\omega_3$
- deformation of hyperkähler metric
- deformation of polynomial invariant $\sim \alpha$