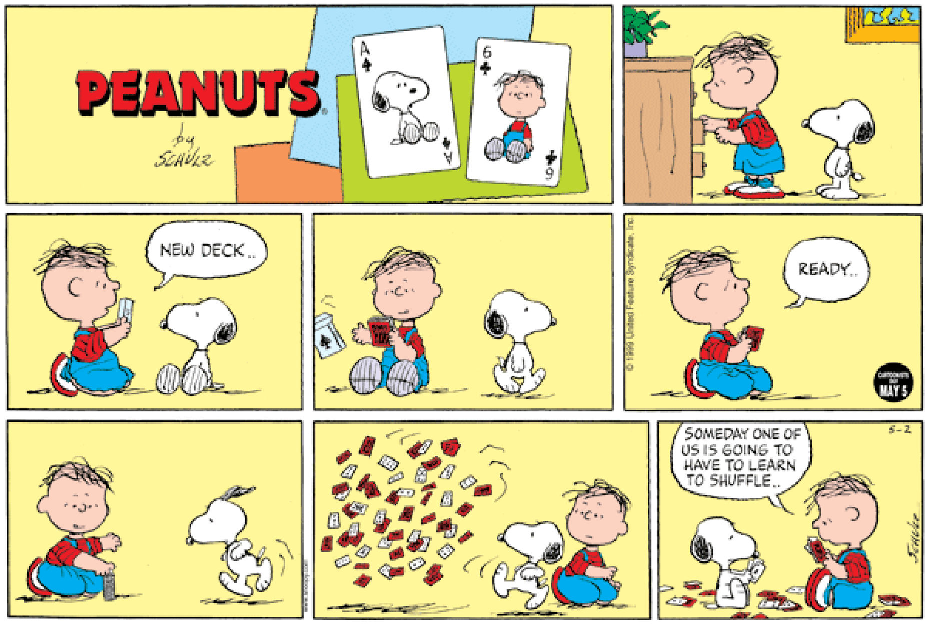
Persi Diaconis
March 2023

**Adding Numbers and Shuffling Cards**




Let $K(i,j)$ be the chance of a carry $j$ following a carry of $i$ when $n$ numbers are added mod $b$; $0 \le i,j \le b-1$.

$$n=2 \qquad K = \frac{1}{2b}\begin{pmatrix} b+1 & b-1 \\ b-1 & b+1 \end{pmatrix}, \qquad i,j \in \{0,1\}$$

$$n=3 \qquad K = \frac{1}{6b^2}\begin{pmatrix} b^2+3b+2 & 4b^2-4 & b^2-3b+2 \\ b^2-1 & 4b^2+2 & b^2-1 \\ b^2-3b+2 & 4b^2-4 & b^2+3b+2 \end{pmatrix}, \qquad i,j \in \{0,1,2\}$$

$$\text{General } n \qquad K(i,j) = \sum_{n=0}^{j-\lfloor i/b \rfloor} (-1)^r \binom{n+1}{r} \binom{n-1-i+(j+1-r)b}{n}, \qquad 0 \le i,j \le b-1$$

$$\text{General } n,\ b=2 \qquad K(i,j) = \frac{1}{2^n}\binom{n+1}{2j-i+1}$$

## References

[1] Dave Bayer and Persi Diaconis. Trailing the dovetail shuffle to its lair. *Ann. Appl. Probab.*, 2(2):294–313, 1992.

[2] John M. Holte. Carries, combinatorics, and an amazing matrix. *Amer. Math. Mon.*, 104(2):138–149, 1997.

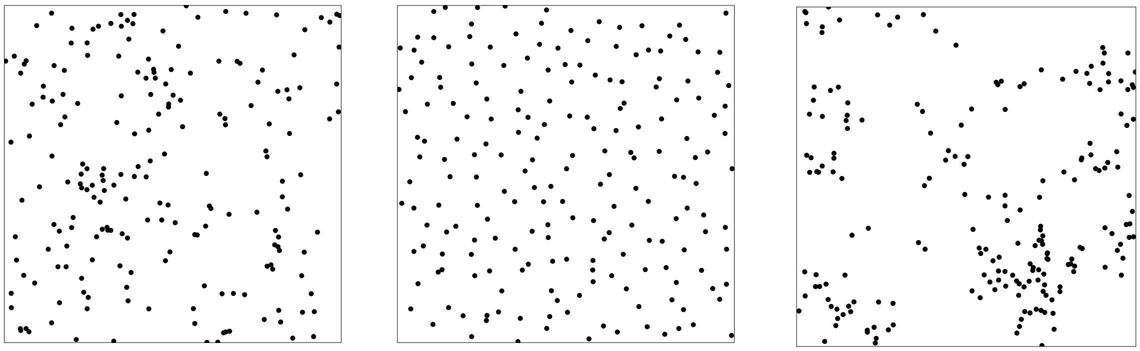
[3] Persi Diaconis and Jason Fulman. Carries, shuffling, and an amazing matrix. *Amer. Math. Mon.*, 116(9):788–803, 2009.

[4] Alexei Borodin, Persi Diaconis, and Jason Fulman. On adding a list of numbers (and other one-dependent determinantal processes). *Bull. Amer. Math. Soc.*, 47(4):639–670, 2010.

[5] Persi Diaconis and Jason Fulman. *The Mathematics of Shuffling Cards.* American Mathematical Society, 2023. To be published.

Persi Diaconis
March 2023

**Adding Numbers and Determinantal Point Processes**



Three samples of translation-invariant point processes in the plane.
Determinantal processes exhibit repulsion while permanental processes exhibit clumping.

## References

Borodin, A. (2009). Determinantal point processes. *arXiv e-prints* doi:10.48550/arXiv.0911.1153, **a good, short, survey**.

Borodin, A., Diaconis, P. and Fulman, J. (2010). On adding a list of numbers (and other one-dependent determinantal processes). *Bull. Amer. Math. Soc.* 47: 639–670, doi:10.1090/S0273-0979-2010-01306-9, **this talk**.

Diaconis, P. and Fulman, J. (2009). Carries, shuffling, and an amazing matrix. *Amer. Math. Mon.* 116: 788–803, doi:10.4169/000298909X474864, **this talk**.

Diaconis, P. and Fulman, J. (2014). Combinatorics of balanced carries. *Adv. in Appl. Math.* 59: 8–25, doi:10.1016/j.aam.2014.05.005, **more signed digits**.

Diaconis, P., Fulman, J. and Holmes, S. (2013). Analysis of casino shelf shuffling machines. *Ann. Appl. Probab.* 23: 1692–1720, doi:10.1214/12-AAP884, **shuffling and signed digits**.

Diaconis, P., Shao, X. and Soundararajan, K. (2014). Carries, group theory and additive combinatorics. *Amer. Math. Monthly* 121: 674–688, doi:10.4169/amer.math.monthly.121.08.674, **the 7/9 theorem**.

Hafiz Affandi, R., Kulesza, A. and Fox, E. B. (2012). Markov determinantal point processes. *arXiv e-prints* doi:10.48550/arXiv.1210.4850, **applications to recommender systems**.

Hough, J. B., Krishnapur, M., Peres, Y. and Virág, B. (2009). *Zeros of Gaussian analytic functions and determinantal point processes*, University Lecture Series *51*. American Mathematical Society, Providence, RI, doi:10.1090/ulect/051, **lots of examples**.

Lyons, R. (2003). Determinantal probability measures. *Publ. Math. Inst. Hautes Études Sci.* 98: 167–212, doi:10.1007/s10240-003-0016-0, **the matroid connection**.

Soshnikov, A. (2000). Determinantal random point fields (translation). *Russian Math. Surveys* 55: 923–975, doi:10.1070/rm2000v055n05ABEH000321, **good standard survey in general spaces**.

TALK 3 SHUFFLING CARDS AND HYPERPLANE ARRANGEMENTS: SOME REFERENCES

1. SOME BASICS THE CONNECTIONS BETWEEN SHUFFLING AND HYPERPLANES WAS DISCOVERED BY PAT BIDIGARE, PHIL HANLON AND DAN ROCKMORE (1999) 'A COMBINATORIAL GENERALIZATION OF THE SPECTRUM OF THE TSETLIN LIBRARY AND ITS GENERALIZATION TO HYPERPLANE ARRANGEMENTS' DUKE MATH. JOURNAL. THE THEORY IS DEVELOPED IN BROWN, K AND DIACONIS, P. (1998) RANDOM WALK AND HYPERPLANE ARRANGEMENTS, ANN. PROBAB.

2. FOR THE GENERALIZATIONS TO SEMIGROUP WALKS, SEE BROWN, K (2000) SEMIGROUPS, RINGS AND MARKOV CHAINS, J. THEORET. PROBAB. AND MARGOLIS, S., SALIOLA, F. AND STEINBERG, B. (2018) CELL COMPLEXES, POSET TOPOLOGY AND THE REPRESENTATION THEORY OF ALGEBRAS ARISING IN ALGEBRAIC COMBINATORICS AND DISCRETE GEOMETRY; THIS IS A BOOK, BUT ALSO ARXIV 1508.05446.

3. FOR THE AMAZING (AND WONDERFUL) GENERALIZATIONS OF 'TYPE A' ALGEBRA TO ALL TYPES DUE TO MARCELLO AGUIAR AND SWAPNEIL MAHAJAN, SEE THE ~~THREE~~ FOUR (LARGE!) BOOKS

- COXETER GROUPS AND HOPF ALGEBRAS A.MS (2006)
- MONOIDAL FUNCTORS, SPECIES AND HOPF ALGEBRAS AMS (2010)
- TOPICS IN HYPERPLANE ARRANGEMENTS AMS (2017)
- BIMONOIDS FOR HYPERPLANE ARRANGEMENTS (2020) CAMBRIDGE